

Local and semilocal Poincaré inequalities on metric spaces

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Abstract. We consider several local versions of the doubling condition and Poincaré inequalities on metric spaces. Our first result is that in proper connected spaces, the weakest local assumptions self-improve to semilocal ones, i.e. holding within every ball.

We then study various geometrical and analytical consequences of such local assumptions, such as local quasiconvexity, self-improvement of Poincaré inequalities, existence of Lebesgue points, density of Lipschitz functions and quasicontinuity of Sobolev functions. It turns out that local versions of these properties hold under local assumptions, even though they are not always straightforward.

We also conclude that many qualitative, as well as quantitative, properties of p -harmonic functions on metric spaces can be proved in various forms under such local assumptions, with the main exception being the Liouville theorem, which fails without global assumptions.

Key words and phrases: capacity, density of Lipschitz functions, Lebesgue point, local doubling, metric space, Newtonian space, nonlinear potential theory, p -harmonic function, Poincaré inequality, quasicontinuity, quasiminimizer, semilocal doubling, Sobolev space.

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1. Introduction

In the last two decades, an extensive part of first-order analysis, such as the Sobolev space theory and nonlinear potential theory for p -harmonic functions, has been developed on metric spaces. Standard assumptions are very often that μ is a doubling measure supporting a p -Poincaré inequality and that the space X is complete. The doubling condition controls changes in scales, while the Poincaré inequality guarantees that functions are controlled by their so-called upper gradients. Both of these conditions play a vital role in many proofs. These assumptions are usually imposed globally on the whole space.

In this paper we study how these conditions can be relaxed and replaced by similar local or semilocal assumptions, while retaining most of the important consequences. We assume throughout the paper that $1 \leq p < \infty$ and that $X = (X, d, \mu)$ is a metric space equipped with a metric d and a positive complete Borel measure μ such that $0 < \mu(B) < \infty$ for all balls $B \subset X$.

Definition 1.1. The measure μ is *locally doubling* if for every $x \in X$ there are $r, C > 0$ (depending on x) such that $\mu(2B) \leq C\mu(B)$ for all balls $B \subset B(x, r)$. If

such a C exists for all $x \in X$ and all $r > 0$, then μ is *semilocally doubling*. (Here and below, λB stands for the ball concentric with B and with λ -times the radius.)

The measure μ supports a *local p -Poincaré inequality* if for every $x \in X$ there are $r, C > 0$ and $\lambda \geq 1$ such that for all balls $B \subset B(x, r)$, all integrable functions u on λB , and all upper gradients g of u ,

$$\int_B |u - u_B| d\mu \leq Cr_B \left(\int_{\lambda B} g^p d\mu \right)^{1/p}. \quad (1.1)$$

If such C and λ exist for all x and all r , then X supports a *semilocal p -Poincaré inequality*.

It may be worth to compare the situation with the definition of local function spaces. A function $f \in L^1_{\text{loc}}(X)$ if for every $x \in X$ there is r such $f \in L^1(B(x, r))$. If X is proper (i.e. all closed bounded sets are compact), then it is a well-known (and useful) fact that this is equivalent to requiring that $f \in L^1_{\text{loc}}(B)$ for every ball B in X . It turns out that a similar fact is true for the doubling property. Note that if μ is globally doubling, then X is proper if and only if it is complete.

Proposition 1.2. *If X is proper and μ is locally doubling, then μ is semilocally doubling.*

This is perhaps not very surprising, and essentially just needs a standard compactness argument to be shown. That a similar result is true also for Poincaré inequalities is far less obvious, and requires several pages to prove. Poincaré inequalities are intimately related to connectivity properties, and thus the precise statement is as follows. The assumptions of properness and connectedness cannot be dropped.

Theorem 1.3. *If X is proper and connected and μ is locally doubling and supports a local p -Poincaré inequality, then it supports a semilocal p -Poincaré inequality.*

As already mentioned, in much of the metric space literature on first-order analysis it is assumed that μ is globally doubling and supports a global p -Poincaré inequality, while it is folklore that much of the theory holds under local assumptions. For instance, the assumptions are global in the monographs Hajlasz–Koskela [22], Björn–Björn [6] and Heinonen–Koskela–Shanmugalingam–Tyson [27]. In [22] and [6] it is mentioned that Riemannian manifolds with Ricci curvature bounded from below support semilocal assumptions with the implicit implication that most of the developed theory holds under these weaker assumptions.

There are some papers requiring local assumptions, but they often take different forms from paper to paper. Sometimes it is assumed that the constants involved are uniform, something we do not assume (but for Section 6). Such uniform local assumptions (of different kinds) are e.g. assumed in Cheeger [17], Danielli–Garofalo–Marola [18], Garofalo–Marola [20] and Holopainen–Shanmugalingam [28]. The requirements are in all cases more restrictive than our local assumptions, and those in [17] are more restrictive than our semilocal assumptions.

Once we have established Proposition 1.2 and Theorem 1.3 (which we do in Sections 3 and 4), we take a look at which useful consequences of global doubling and Poincaré inequalities can be obtained already under (semi)local assumptions, with and without properness. We study the self-improvement of Poincaré inequalities, Lebesgue points, density of Lipschitz and locally Lipschitz functions, quasicontinuity and p -harmonic functions under (semi)local assumptions.

In Section 5 we concentrate on the self-improvement of Poincaré inequalities and prove two results: one improving the norm in the left-hand side of (1.1) and the other one the norm in the right-hand side. In Section 6 we see how local

uniform assumptions give slightly stronger self-improvement conclusions. Under global assumptions these important results are due to Hajlasz–Koskela [21], [22] and Keith–Zhong [31], respectively. In particular, the latter result can be localized in the following way.

Theorem 1.4. *If X is locally compact and μ is locally doubling and supports a local p -Poincaré inequality, both with uniform constants C and λ and with $p > 1$, then X supports a local q -Poincaré inequality for some $q < p$ with new uniform constants C and λ .*

It is worth noting that the corresponding result with global assumptions and global conclusion fails in locally compact spaces, due to a counterexample by Koskela [37]. In complete spaces it holds by Keith–Zhong [31].

In Section 7, we turn to Lebesgue points and show that Sobolev (Newtonian) functions have Lebesgue points outside a set of zero p -capacity. Traditionally, as well as in metric spaces, such results are shown using the density of continuous functions. Here we avoid using this property and instead exploit the Newtonian theory in a different and novel way, which may be of interest also under global assumptions.

In the next section we consider the density of Lipschitz and locally Lipschitz functions in the Sobolev (Newtonian) space $N^{1,p}(X)$. There are two existing results under global assumptions in the literature, one assuming doubling and a Poincaré inequality, due to Shanmugalingam [40], and the other more recent one assuming $p > 1$, completeness and doubling but no Poincaré inequality, due to Ambrosio–Colombo–Di Marino [3] and Ambrosio–Gigli–Savaré [4]. We extend both results to local assumptions, and combine them. Among other results, we obtain the following “local-to-global” density result.

Theorem 1.5. *If X is proper and connected and μ is locally doubling and supports a local p -Poincaré inequality then Lipschitz functions with compact support are dense in $N^{1,p}(X)$.*

In Section 9 we look at consequences of the obtained density results, primarily quasicontinuity and various properties of the Sobolev capacity C_p . We end the paper with a discussion on how much of the nonlinear potential theory for p -harmonic functions, developed e.g. in the book [6], holds under local assumptions, and explain that indeed most of the results therein extend to this setting, with the main exception being the Liouville theorem, which actually fails without global assumptions. As most of the results in [6] are either local or semilocal (e.g. on bounded domains) this is not so surprising, and indeed this has already been hinted upon in the literature, as mentioned above.

The importance of distinguishing between local and global assumptions is certainly more apparent when discussing global properties, such as the Dirichlet problem on unbounded domains (as in Hansevi [23], [24]) or existence of global singular functions (as in Holopainen–Shanmugalingam [28]). We hope that the theory developed in this paper will provide a suitable foundation for such studies.

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2. Upper gradients and Newtonian spaces

We assume throughout the paper that $1 \leq p < \infty$ and that $X = (X, d, \mu)$ is a metric space equipped with a metric d and a positive complete Borel measure μ such that $0 < \mu(B) < \infty$ for all balls $B \subset X$. It follows that X is separable and Lindelöf.

Proofs of the results in this section can be found in the monographs Björn–Björn [6] and Heinonen–Koskela–Shanmugalingam–Tyson [27].

A *curve* is a continuous mapping from an interval, and a *rectifiable* curve is a curve with finite length. We will only consider curves which are compact and rectifiable, and thus each curve can be parameterized by its arc length ds . A property is said to hold for *p -almost every curve* if it fails only for a curve family Γ with zero p -modulus, i.e. there exists $0 \leq \rho \in L^p(X)$ such that $\int_\gamma \rho ds = \infty$ for every rectifiable curve $\gamma \in \Gamma$.

Following Heinonen–Koskela [26], we introduce upper gradients as follows (they called them very weak gradients).

Definition 2.1. A Borel function $g : X \rightarrow [0, \infty]$ is an *upper gradient* of a function $f : X \rightarrow \overline{\mathbf{R}} := [-\infty, \infty]$ if for all nonconstant rectifiable curves $\gamma : [0, l_\gamma] \rightarrow X$,

$$|f(\gamma(0)) - f(\gamma(l_\gamma))| \leq \int_\gamma g ds, \quad (2.1)$$

where the left-hand side is considered to be ∞ whenever at least one of the terms therein is infinite. If $g : X \rightarrow [0, \infty]$ is measurable and (2.1) holds for p -almost every nonconstant rectifiable curve, then g is a *p -weak upper gradient* of f .

The p -weak upper gradients were introduced in Koskela–MacManus [38]. It was also shown therein that if $g \in L^p_{\text{loc}}(X)$ is a p -weak upper gradient of f , then one can find a sequence $\{g_j\}_{j=1}^\infty$ of upper gradients of f such that $\|g_j - g\|_{L^p(X)} \rightarrow 0$. If f has an upper gradient in $L^p_{\text{loc}}(X)$, then it has an a.e. unique *minimal p -weak upper gradient* $g_f \in L^p_{\text{loc}}(X)$ in the sense that for every p -weak upper gradient $g \in L^p_{\text{loc}}(X)$ of f we have $g_f \leq g$ a.e., see Shanmugalingam [41]. Following Shanmugalingam [40], we define a version of Sobolev spaces on the metric space X .

Definition 2.2. For a measurable function $f : X \rightarrow \overline{\mathbf{R}}$, let

$$\|f\|_{N^{1,p}(X)} = \left(\int_X |f|^p d\mu + \inf_g \int_X g^p d\mu \right)^{1/p},$$

where the infimum is taken over all upper gradients g of f . The *Newtonian space* on X is

$$N^{1,p}(X) = \{f : \|f\|_{N^{1,p}(X)} < \infty\}.$$

The quotient space $N^{1,p}(X)/\sim$, where $f \sim h$ if and only if $\|f - h\|_{N^{1,p}(X)} = 0$, is a Banach space and a lattice, see Shanmugalingam [40]. In this paper we assume that functions in $N^{1,p}(X)$ are defined everywhere (with values in $\overline{\mathbf{R}}$), not just up to an equivalence class in the corresponding function space. This is important for upper gradients to make sense.

For a measurable set $E \subset X$, the Newtonian space $N^{1,p}(E)$ is defined by considering $(E, d|_E, \mu|_E)$ as a metric space in its own right. We say that $f \in N^{1,p}_{\text{loc}}(E)$ if for every $x \in E$ there exists a ball $B_x \ni x$ such that $f \in N^{1,p}(B_x \cap E)$. If $f, h \in N^{1,p}_{\text{loc}}(X)$, then $g_f = g_h$ a.e. in $\{x \in X : f(x) = h(x)\}$, in particular for $c \in \mathbf{R}$ we have $g_{\min\{f,c\}} = g_f \chi_{\{f < c\}}$ a.e.

Definition 2.3. The (Sobolev) *capacity* of a set E is the number

$$C_p(E) = \inf_u \|u\|_{N^{1,p}(X)}^p,$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u = 1$ on E .

We say that a property holds *quasieverywhere* (q.e.) if the set of points for which the property does not hold has capacity zero. The capacity is the correct gauge for distinguishing between two Newtonian functions. If $u \in N^{1,p}(X)$, then $u \sim v$ if and only if $u = v$ q.e. Moreover, if $u, v \in N_{\text{loc}}^{1,p}(X)$ and $u = v$ a.e., then $u = v$ q.e.

We let $B = B(x, r) = \{y \in X : d(x, y) < r\}$ denote the ball with centre x and radius r , and let $\lambda B = B(x, \lambda r)$. We assume throughout the paper that balls are open. In metric spaces it can happen that balls with different centres and/or radii denote the same set. We will however make the convention that a ball B comes with a predetermined centre and radius r_B . Note that it can happen that $B(x_0, r_0) \subset B(x_1, r_1)$ even when $r_0 > r_1$. In disconnected spaces this can happen also when $r_0 > 2r_1$. If X is connected, then $r_0 > 2r_1$ is possible only when $B(x_0, r_0) = B(x_1, r_1) = X$.

3. Local doubling

One can think of several different possibilities for local assumptions. We will make them precise below. In this section we concentrate on the doubling property and then consider Poincaré inequalities in the next section.

Definition 3.1. The measure μ is *doubling within* $B(x_0, r_0)$ if there is $C > 0$ (depending on x_0 and r_0) such that $\mu(2B) \leq C\mu(B)$ for all balls $B \subset B(x_0, r_0)$.

We say that μ is *locally doubling* (on X) if for every $x_0 \in X$ there is $r_0 > 0$ (depending on x_0) such that μ is doubling within $B(x_0, r_0)$.

If μ is doubling within every ball $B(x_0, r_0)$ then it is *semilocally doubling* (on X), and if moreover C is independent of x_0 and r_0 , then μ is *globally doubling* (on X).

Note that when saying that μ is doubling *within* $B(x_0, r_0)$ this is (implicitly) done with respect to X as the balls are all with respect to X , and moreover $2B$ does not have to be a subset of $B(x_0, r_0)$. This is not the same as saying that μ is globally doubling *on* $B(x_0, r_0)$, which refers to balls with respect to $B(x_0, r_0)$.

If μ is locally doubling on X and $\Omega \subset X$ is open, then μ is also locally doubling on Ω . This fails for semilocal and global doubling, see Remark 3.3 below.

Another related property is that μ is *pointwise doubling* at $x_0 \in X$ if there are $C, r_0 > 0$ such that $\mu(B(x_0, 2r)) \leq C\mu(B(x_0, r))$ for $0 < r < r_0$. Requiring such a pointwise assumption and a similar pointwise Poincaré inequality at every $x_0 \in X$ is too weak for most results. See however, Björn–Björn–Lehrbäck [8], [9] for some results using such pointwise assumptions.

Definition 3.2. The space X is *globally doubling* if there is a constant N such that every ball $B(x, r)$ can be covered by at most N balls with radii $\frac{1}{2}r$.

The space X is *locally doubling* if for every $x_0 \in X$ there is $r_0 > 0$ such that $B(x_0, r_0)$ is globally doubling. Moreover, X is *semilocally doubling* if every ball $B \subset X$ is globally doubling.

Remark 3.3. Let B be a ball. If μ is globally doubling then it does not follow that $\mu|_B$ is globally doubling on B , see Example 4.3. On the other hand if the space X is globally doubling then so is B (as a metric space). This is why Definition 3.2 differs from Definition 3.1 in that it considers balls with respect to B which are not balls with respect to X . It is possible to give an equivalent definition of (semi)local doubling of X more in the spirit of Definition 3.1, which only uses balls with respect to X , but such a definition is more technical to state and hence we prefer our Definition 3.2.

It is rather immediate that every subset of a globally doubling metric space is itself globally doubling, and hence the same hereditary property also holds for (semi)local doubling. It is also easy to see that every bounded set in a semilocally doubling metric space is totally bounded. See Heinonen [25, Section 10.13] for more on doubling metric spaces.

If μ is (semi)locally resp. globally doubling, then so is X by the following result.

Proposition 3.4. *Assume that μ is doubling within $B_0 = B(x_0, r_0)$ in the sense of Definition 3.1. Then δB_0 is globally doubling for every $\delta < \frac{2}{3}$, with N depending only on δ and the doubling constant within B_0 .*

Example 3.5 below shows that the constant $\frac{2}{3}$ is optimal and that it can even happen that $\frac{2}{3}B_0$ is not totally bounded.

Proof. Let $B' = B(x, r) \cap \delta B_0$ be an arbitrary ball with respect to δB_0 for some $\delta < \frac{2}{3}$. Then $x \in \delta B_0$ and we may assume that $r \leq 2\delta r_0$. Let $B = B(x, r)$ and $r' = \min\{\frac{1}{4}r, \frac{1}{24}r_0\}$. Assume that $x_i \in B'$, $i = 1, \dots, N$, are such that $d(x_i, x_j) \geq 2r'$ if $i \neq j$. Then $B(x_i, r')$ are pairwise disjoint and $B(x_i, 8r') \subset B_0$. We shall show that there is a bound for N . Let C_μ be the doubling constant for μ within B_0 .

If $r' = \frac{1}{4}r$, then

$$\mu(2B) \leq \mu(B(x_i, 16r')) \leq C_\mu \mu(B(x_i, 8r')) \leq C_\mu^4 \mu(B(x_i, r')), \quad (3.1)$$

and hence

$$N \min_i \mu(B(x_i, r')) \leq \sum_{i=1}^N \mu(B(x_i, r')) \leq \mu(2B) \leq C_\mu^4 \min_i \mu(B(x_i, r')),$$

which yields that $N \leq C_\mu^4$. On the other hand, if $r' = \frac{1}{24}r_0$ then as in (3.1),

$$\mu\left(\left(\frac{2}{3} - \delta\right)B_0\right) \leq \mu\left(B\left(x_i, \frac{2}{3}r_0\right)\right) = \mu(B(x_i, 16r')) \leq C_\mu^4 \mu(B(x_i, r')).$$

Hence

$$\begin{aligned} N \min_i \mu(B(x_i, r')) &\leq \sum_{i=1}^N \mu(B(x_i, r')) \leq \mu(B_0) \\ &\leq \frac{C_\mu^4 \mu(B_0)}{\mu\left(\left(\frac{2}{3} - \delta\right)B_0\right)} \min_i \mu(B(x_i, r')) \leq M \min_i \mu(B(x_i, r')), \end{aligned}$$

where M only depends on C_μ and δ . Hence $N \leq M$.

We can thus find a maximal pairwise disjoint collection of $\{B(x_i, r')\}_{i=1}^N$ with $N \leq \max\{M, C_\mu^4\}$ elements. As the collection is maximal we see that

$$B' \subset \bigcup_{i=1}^N B(x_i, 2r') \subset \bigcup_{i=1}^N B(x_i, \frac{1}{2}r).$$

Hence δB_0 is globally doubling. \square

Example 3.5. Let $I_0 = \{0\} \times [-1, 0] \subset \mathbf{R}^2$ and $I_j = \{j\} \times [\frac{2}{3} - \frac{1}{3} \cdot 2^{-j}, 1] \subset \mathbf{R}^2$, $j = 1, 2, \dots$, be vertical linear segments in the plane. Equip each I_j with a multiple of the 1-dimensional Hausdorff measure so that $\mu(I_j) = 2^{-j}$, $j = 0, 1, \dots$. Let $X = \bigcup_{j=0}^{\infty} I_j$, equipped with μ and the metric d so that for $x = (j, x_2) \in I_j$ and $y = (k, y_2) \in I_k$,

$$d(x, y) = \begin{cases} |x_2 - y_2|, & \text{if } j = k \text{ or } j = 0 \text{ or } k = 0, \\ 1, & \text{if } j \neq k, j, k \geq 1. \end{cases}$$

Let $x_0 = (0, 0)$. Then it is easily verified that μ is doubling within $B_0 = B(x_0, 1)$. However, $\frac{2}{3}B_0$ is not totally bounded, and thus not globally doubling.

This example also shows that the next two results are sharp. More precisely, X is bounded and complete but noncompact and thus not proper. Both X and μ are locally doubling but neither is semilocally doubling.

The most common global assumptions are that X is complete and supports a global p -Poincaré inequality and that μ is globally doubling. It then follows that X is proper and connected (and even quasiconvex). Under local assumptions these properties need to be imposed separately. Connectedness is strongly related to Poincaré inequalities, which we discuss in the next section. Properness always implies completeness and the proof of [6, Proposition 3.1] also shows the following equivalence.

Lemma 3.6. *Assume that X is semilocally doubling. Then X is proper if and only if X is complete.*

If X is only locally doubling and complete, then it is locally compact but not necessarily proper as the following example shows.

Example 3.7. Let $X = \mathbf{R}^2$ equipped with the Gaussian measure $Ce^{-x_1^2 - x_2^2} dx$ but with the distance $d(x, y) = \arctan|x - y|$. Then X is bounded and complete, but not proper (as X is a closed bounded noncompact set). At the same time, μ clearly is locally doubling and supports a local 1-Poincaré inequality.

In proper spaces, local and semilocal doubling are equivalent, as we shall now see.

Proposition 3.8. *Assume that X is proper. If X resp. μ is locally doubling, then it is semilocally doubling.*

Proof. Let B_0 be a ball. If X is locally doubling, we can for each $x \in \overline{B_0}$ find a globally doubling ball $B_x \ni x$. Since X is proper, $\overline{B_0}$ is compact and thus we can find a finite set $\{x_i\}_{i=1}^N$ such that $B_0 \subset \bigcup_{i=1}^N B_{x_i}$. It is easily seen that a finite union of globally doubling sets is globally doubling, and hence B_0 is globally doubling.

Now assume instead that μ is locally doubling and $B_0 = B(x_0, r_0)$. By enlarging r_0 , if necessary, we may assume that either $r_0 = \text{dist}(x_0, X \setminus B_0)$ or $B_0 = X$. Since $\overline{B_0}$ is compact (as X is proper), it can be covered by finitely many balls $B_j = B(x_j, r_j)$ such that μ is doubling within each ball $2B_j$. Let $r' = \min_j r_j$. Again by compactness, we can cover $\overline{B_0}$ by finitely many balls B'_j with radii $r'/2$. Let $B(x, r) \subset B_0$ be arbitrary and find j and j' such that $x \in B_j$ and $x \in B'_{j'}$.

If $r \leq r'$ then $B(x, r) \subset 2B_j$ and hence the conclusion of the proposition holds for $B(x, r)$ with constant $\max_j C_j$. On the other hand, if $r > r'$ then $x \in B'_{j'}$ and hence $B'_{j'} \subset B(x, r)$, which yields

$$\mu(B(x, r)) \geq \min_j \mu(B'_j) > 0.$$

Since $B(x, 2r) \subset B(x_0, 5r_0)$, we also have a uniform upper bound for $\mu(B(x, 2r))$, which proves that μ is semilocally doubling. \square

We end the section by noting the following consequence of local doubling, which will be useful later.

Theorem 3.9. (The Lebesgue differentiation theorem) *Assume that μ is locally doubling. If $f \in L^1(X)$, then a.e. point is a Lebesgue point for f .*

Proof. As X is Lindelöf, we can cover X by balls $\{B_j\}_{j=1}^\infty$ such that μ is doubling within each $10B_j$. By the proof of Theorem 1.6 in Heinonen [25], the Vitali covering theorem holds in each B_j . It then follows from Remark 1.13 in [25] that the Lebesgue differentiation theorem holds within each B_j , and hence in X , as the union is countable. \square

4. Local Poincaré inequalities

In this section we study local aspects of the Poincaré inequality similarly to the doubling property in Section 3. It will turn out that connectivity plays an important role here.

Definition 4.1. Let $1 \leq q < \infty$. We say that the (q, p) -Poincaré inequality holds within $B(x_0, r_0)$ if there are constants $C > 0$ and $\lambda \geq 1$ (depending on x_0 and r_0) such that for all balls $B \subset B(x_0, r_0)$, all integrable functions u on λB , and all upper gradients g of u ,

$$\left(\int_B |u - u_B|^q d\mu \right)^{1/q} \leq C r_B \left(\int_{\lambda B} g^p d\mu \right)^{1/p}. \quad (4.1)$$

We also say that X (or μ) supports a *local* (q, p) -Poincaré inequality (on X) if for every $x_0 \in X$ there is r_0 (depending on x_0) such that the (q, p) -Poincaré inequality holds within $B(x_0, r_0)$.

If the (q, p) -Poincaré inequality holds within every ball $B(x_0, r_0)$ then X supports a *semilocal* (q, p) -Poincaré inequality, and if moreover C and λ are independent of x_0 and r_0 , then X supports a *global* (q, p) -Poincaré inequality.

If $q = 1$ we usually just write *p-Poincaré inequality*.

The inequality (4.1) can equivalently be required for all integrable functions u on λB , and all p -weak upper gradients g of u .

As in the case of the doubling condition, local Poincaré inequalities are inherited by open subsets, i.e. if $\Omega \subset X$ is open and X supports a local (q, p) -Poincaré inequality, then so does Ω . This fails for semilocal and global Poincaré inequalities.

Remark 4.2. When defining (semi)local doubling in Definition 3.1 it is primarily a matter of taste (except for the constant $\frac{2}{3}$ in Proposition 3.4) whether the condition is required for $B \subset B(x_0, r_0)$ or for $B \subset 2B \subset B(x_0, r_0)$, and the same is true for B and λB in the local Poincaré inequalities in Definition 4.1. However, for semilocal Poincaré inequalities it is vital that the condition is for all $B \subset B_0$, rather than for all $B \subset \lambda B \subset B_0$, which is a weaker requirement since λ is allowed to depend on B_0 . (Consider e.g. $X = (-\infty, 0] \cup [1, \infty)$ with the Euclidean metric and Lebesgue measure. Then for $x_0 = 0$, $r_0 \geq 2$ and $\lambda := r_0$, the requirement $\lambda B \subset B(x_0, r_0)$ implies that $r_B \leq 1$ and hence (4.1) holds for all such balls, while it clearly fails for $B(0, 2) \subset B(x_0, r_0)$.)

Example 4.3. If a Poincaré inequality holds within $B_0 = B(x_0, r_0)$, then it does not mean that B_0 (or \overline{B}_0) itself (as a metric space) supports a global Poincaré inequality. Similarly, if μ is doubling within B_0 , then it does not follow that $\mu|_{B_0}$ (or $\mu|_{\overline{B}_0}$) is doubling. To see this, let

$$X = \mathbf{R}^2 \setminus \{(x, y) \in \mathbf{R}^2 : 0 < y < \sqrt{1 - x^2} - e^{-1/|x|}\},$$

i.e. X is \mathbf{R}^2 with the open unit upper half-disc removed and the curved cusp of exponential type at $(0, 1)$,

$$C_0 = \{(x, y) \in \mathbf{R}^2 : 0 < \sqrt{1 - x^2} - e^{-1/|x|} \leq y < \sqrt{1 - x^2}\},$$

inserted back into the hole. Then $\text{int } X$ is a uniform domain and hence the Lebesgue measure, restricted to it, is globally doubling and supports a global 1-Poincaré inequality, by Theorem 4.4 in Björn–Shanmugalingam [15] (or [6, Theorem A.21]). By Aikawa–Shanmugalingam [2, Proposition 7.1], the same is true for X itself.

Now, if x_0 is the origin and we let $B_0 = B(x_0, 1)$, then $B_0 \cap X$ and $\overline{B_0 \cap X}$ have the cusps C_0 and $\overline{C_0}$ in the vicinity of the point $(0, 1)$. Hence the Lebesgue measure restricted to these sets is not doubling near or at this point.

Moreover, C_0 is disconnected at $(0, 1)$ and $\overline{C_0}$ is essentially disconnected at the point $(0, 1)$ (since it has zero capacity with respect to $\overline{B_0 \cap X}$ for all $p \geq 1$), so neither $B_0 \cap X$ nor $\overline{B_0 \cap X}$ supports any global or semilocal Poincaré inequalities. For $\overline{B_0 \cap X}$ even the local doubling and all local Poincaré inequalities fail at $(0, 1)$.

Proposition 3.8 about (semi)local doubling is rather straightforward. A bit more surprising, perhaps, is that Theorem 1.3 is true for Poincaré inequalities. We will obtain the following more general version of Theorem 1.3.

Theorem 4.4. *If X is proper and connected and μ is locally doubling and supports a local (q, p) -Poincaré inequality, then it supports a semilocal (q, p) -Poincaré inequality.*

The proof of Theorem 4.4 will be split into a number of lemmas, some of which may be of independent interest. It will be concluded at the end of this section. In the meantime, we discuss the assumptions in Theorem 4.4 as well as some consequences of local Poincaré inequalities. To start with, it is easily verified that if X supports a semilocal p -Poincaré inequality then it is connected, cf. the proof of [6, Proposition 4.2]. The following example shows that this conclusion fails if we replace the semilocal assumption with a local one, even if X is proper.

Example 4.5. Let X be the union of two disjoint closed balls in \mathbf{R}^n , which is proper and such that the Lebesgue measure is globally doubling and supports a local 1-Poincaré inequality on X . However X is not connected.

The above example also shows that the connectedness assumption in Theorem 4.4 cannot be dropped, while the following example shows that neither can the properness assumption.

Example 4.6. Let

$$X = (\overline{B(0, 2)} \setminus B(0, 1)) \cup \{x = (x_1, x_2) : 0 < |x| \leq 2 \text{ and } x_1 x_2 \geq 0\} \subset \mathbf{R}^2.$$

Then X is connected and the Lebesgue measure is globally doubling on X and supports a local 1-Poincaré inequality. However, X is neither proper nor supports any semilocal Poincaré inequality.

The following is a partial result on the way to proving Theorem 4.4.

Lemma 4.7. *If X is proper and supports a local (q, p) -Poincaré inequality, then for every ball B_0 there exist $r', C > 0$ and $\lambda \geq 1$ such that the (q, p) -Poincaré inequality (4.1) holds for all balls $B = B(x, r) \subset B_0$ with $r \leq r'$.*

Proof. As in the proof of Proposition 3.8, the compact set $\overline{B_0}$ can be covered by finitely many balls $B_j = B(x_j, r_j)$ so that the (q, p) -Poincaré inequality holds within each $2B_j$, with constants C_j and λ_j . Letting

$$r' = \min_j r_j, \quad C = \max_j C_j \quad \text{and} \quad \lambda = \max_j \lambda_j,$$

together with the fact that $B(x, r) \subset 2B_j$ for some j , concludes the proof. \square

We shall now see which connectivity properties follow from the local Poincaré inequality.

Proposition 4.8. *Assume that X is locally compact and that μ is locally doubling and supports a local p -Poincaré inequality. Then X is locally quasiconvex (and thus locally rectifiably pathconnected), i.e. for every $x_0 \in X$ there are $r_0, L > 0$ (depending on x_0) such that every pair of points $x, y \in B(x_0, r_0)$ can be connected by a curve of length at most $Ld(x, y)$.*

If X is moreover proper and connected, then it is semilocally quasiconvex, i.e. the above connectivity property holds in every ball $B(x_0, r_0)$ (with L depending on it).

Local quasiconvexity and its behaviour under various transformations of X were considered by Buckley–Herron–Xie [16], cf. Section 2 and Proposition 4.2 therein. Note that if X is connected and locally (rectifiably) pathconnected, then it is (rectifiably) pathconnected, as the (rectifiably) pathconnected components must be open. In particular, local quasiconvexity and connectedness imply that X is rectifiably pathconnected, but not necessarily quasiconvex.

Later on it will be important for us to have the following more precise version of the above connectivity result, which will also be used to deduce Proposition 4.8.

Lemma 4.9. *Let $x, y \in X$ and assume that the p -Poincaré inequality and the doubling property for μ hold (with constants C_{PI} and C_μ) within $B_0 = B(x, 2d(x, y))$ in the sense of Definitions 3.1 and 4.1. Let $\Lambda = 8C_\mu^3 C_{\text{PI}}$. If the ball $\overline{\Lambda B_0}$ is compact then x and y can be connected (in X) by a curve of length at most $Ld(x, y)$, where $L = 24\Lambda$.*

Note that, as in [6, Theorem 4.32], the constants Λ and L are independent of the dilation constant λ in the p -Poincaré inequality.

Proof. Following Semmes’s chaining argument, define for $\varepsilon > 0$ and $z \in \lambda B_0$,

$$\rho_\varepsilon(z) = \inf \sum_{i=1}^m d(x_{i-1}, x_i),$$

where λ is the dilation constant in the p -Poincaré inequality within B_0 and the infimum is taken over all collections $\{x_i\}_{i=0}^m \subset X$ such that $x_0 = x$, $x_m = z$ and $d(x_{i-1}, x_i) < \varepsilon$ for all $i = 1, 2, \dots, m$. Then ρ_ε is locally 1-Lipschitz and has 1 as an upper gradient.

Since the p -Poincaré inequality and the doubling property for μ hold for all the balls $B(x, 2d(x, y))$, $B(x, 2^{-j}d(x, y))$ and $B(y, 2^{-j}d(x, y))$, with $j = 1, 2, \dots$, the arguments in the proof of [6, Theorem 4.32] show that $\rho_\varepsilon(y) \leq 2\Lambda d(x, y)$. In particular, for each ε there is a collection $\{x_i\}_{i=0}^m$ with $x_0 = x$ and $x_m = y$ such that $\sum_{i=1}^m d(x_{i-1}, x_i) \leq 3\Lambda d(x, y)$ and all x_i belong to the compact set $\overline{\Lambda B_0}$. As in the proof of [6, Theorem 4.32], this collection can be used to construct a $24\Lambda d(x, y)$ -Lipschitz map from a dyadic partition of $[0, 1]$ into X , and the existence of a desired connecting curve then follows from the Ascoli theorem, upon letting $\varepsilon \rightarrow 0$. \square

Proof of Proposition 4.8. Let $x_0 \in X$ and find a ball B'_0 centred at x_0 so that $\overline{B'_0}$ is compact and the p -Poincaré inequality and the doubling property for μ hold within B'_0 , with constants C_μ and C_{PI} . Let $\Lambda = 8C_\mu^3 C_{\text{PI}}$ and $B_0 = \Lambda^{-1}B'_0$. Now, if $x, y \in \frac{1}{5}B_0$ then $B(x, 2d(x, y)) \subset B_0$ and $\overline{\Lambda B_0}$ is compact. Hence Lemma 4.9 shows the existence of a connecting curve of length at most $24\Lambda d(x, y)$. Thus X is locally quasiconvex, which proves the first part of the proposition.

Now assume that X is in addition proper and connected. Let $B_0 = B(x_0, r_0)$ be arbitrary. By Proposition 3.8, μ is semilocally doubling. Furthermore, Lemma 4.7 implies that for some $0 < r' \leq r_0$, the p -Poincaré inequality (1.1) holds for all $B = B(x, r) \subset 5B_0$ with $0 < r \leq 2r'$.

Since $\overline{5\lambda B_0}$ is compact, by the properness of X , Lemma 4.9 yields that every pair of points $x, y \in \overline{B_0}$ with $d(x, y) \leq r'$, can be connected by a curve of length at most $L'd(x, y)$, where L' depends only on the doubling and Poincaré constants provided for $5B_0$ by Proposition 3.8 and Lemma 4.7.

By compactness, $\overline{B_0}$ can be covered by finitely many balls $B(x_k, r')$ with $x_k \in \overline{B_0}$, $k = 1, 2, \dots, N$. As X is connected and locally quasiconvex, it is rectifiably connected. Hence, there is for each pair x_k, x_j , $j \neq k$, a rectifiable curve $\gamma_{j,k} \subset X$ connecting x_k to x_j . Since there are only finitely many such pairs, it follows that $M := \sup_{j,k} l_{\gamma_{j,k}} < \infty$.

Finally, let $x, y \in B_0$ be arbitrary. If $d(x, y) \leq r'$, then we already know that x and y can be connected by a curve of length at most $L'd(x, y)$. If $d(x, y) > r'$, then x and y can be connected to some x_j and x_k , respectively, by curves of lengths at most $L'r'$. Adding $\gamma_{j,k}$ to these curves produces a curve from x to y of length at most $2L'r' + M < (2L' + M/r')d(x, y)$. \square

The local quasiconvexity proved in Proposition 4.8 can be further bootstrapped by the following result.

Lemma 4.10. *Let B_0 be a ball such that μ is locally doubling and supports a local p -Poincaré inequality on B_0 (as the underlying space). Assume that the p -Poincaré inequality (1.1) holds for $B = B_0$ with dilation $\lambda \geq 1$ and that λB_0 is locally compact.*

Then B_0 is rectifiably connected within λB_0 , i.e. any two points $x, y \in B_0$ can be connected by a rectifiable curve lying within λB_0 .

Proof. Divide λB_0 into its rectifiable components, i.e. $x, y \in \lambda B_0$ belong to the same rectifiable component if there is a rectifiable curve within λB_0 from x to y . For each $x \in \lambda B_0$, let G_x denote the component containing x , which is measurable by Corollary 1.9 and Remark 3.1 in Järvenpää–Järvenpää–Rogovin–Rogovin–Shanmugalingam [29], since λB_0 is locally compact. The local assumptions on B_0 , together with Proposition 4.8, imply that the sets $G_x \cap B_0$ are open for all x .

Let G be such a rectifiable component intersecting B_0 . We shall show that $B_0 \subset G$. Assume on the contrary that $B_0 \setminus G \neq \emptyset$. Since all $G_x \cap B_0$ are open, so is $B_0 \setminus G = \bigcup_{x \notin G} (G_x \cap B_0)$. Let $u = \chi_G$ which has $g \equiv 0$ as an upper gradient in the open set λB_0 as there are no curves in λB_0 from G to $\lambda B_0 \setminus G$. Since both $B_0 \cap G$ and $B_0 \setminus G$ are nonempty and open, they both have positive measure. Hence, by the p -Poincaré inequality for B_0 ,

$$0 < \int_{B_0} |u - u_{B_0}| d\mu \leq Cr_{B_0} \left(\int_{\lambda B_0} g^p d\mu \right)^{1/p} = 0,$$

a contradiction. \square

The following lemma makes it possible to lift the Poincaré inequality from small to large sets.

Lemma 4.11. *Let $1 \leq q < \infty$ and $A, E \subset X$ be such that $\mu(A \cap E) \geq \theta\mu(E)$ for some $\theta > 0$. Also assume that for some $Q \geq 0$ and a measurable function u ,*

$$\|u - u_A\|_{L^q(A)} \leq Q \quad \text{and} \quad \|u - u_E\|_{L^q(E)} \leq Q.$$

Then

$$\|u - u_{A \cup E}\|_{L^q(A \cup E)} \leq 4(1 + \theta^{-1/q})Q.$$

Proof. To start with, we have by the triangle inequality,

$$\begin{aligned} |u_A - u_E| &= \frac{\|u_A - u_E\|_{L^q(A \cap E)}}{\mu(A \cap E)^{1/q}} \\ &\leq \frac{\|u - u_A\|_{L^q(A)} + \|u - u_E\|_{L^q(E)}}{\mu(A \cap E)^{1/q}} \leq \frac{2Q}{\mu(A \cap E)^{1/q}}. \end{aligned}$$

This yields

$$\begin{aligned} \|u - u_A\|_{L^q(A \cup E)} &\leq \|u - u_A\|_{L^q(A)} + \|u - u_E\|_{L^q(E)} + \mu(E)^{1/q} |u_A - u_E| \\ &\leq 2Q + 2Q \left(\frac{\mu(E)}{\mu(A \cap E)} \right)^{1/q} \leq 2(1 + \theta^{-1/q})Q. \end{aligned}$$

Finally, a standard argument allows us to replace u_A in the left-hand side by $u_{A \cup E}$ at the cost of an extra factor 2 in the right-hand side, cf. [6, Lemma 4.17]. \square

We are now ready to conclude the proof of the semilocal (q, p) -Poincaré inequality.

Proof of Theorem 4.4. Let $B_0 = B(x_0, r_0)$ be fixed. As X is proper and connected, Proposition 4.8 shows that there is $\sigma \geq 1$ such that every pair of points in \overline{B}_0 can be connected by a rectifiable curve within σB_0 . By Lemma 4.7, there exist $r', C > 0$ and $\lambda \geq 1$ such that the (q, p) -Poincaré inequality (4.1) holds for every ball B with radius $r_B \leq r'$ and centre in $\overline{\sigma B}_0$. By decreasing r' , if necessary, we may assume that $\lambda r' \leq r_0$.

Next, let $B \subset B_0$ be an arbitrary ball. If $r_B \leq r'$, then the (q, p) -Poincaré inequality (4.1) holds for B . Assume therefore that $r_B > r'$. By compactness, \overline{B}_0 can be covered by finitely many balls B'_j , $j = 1, 2, \dots, M$, with radius r' and centres $x_j \in \overline{B}_0$. Proposition 4.8 provides us for each j with a rectifiable curve γ_j in σB_0 connecting x_j to x_{j+1} . Following this curve, we define balls

$$B_{j,k} = B(\gamma_j(kr'/2), r'), \quad k = 1, 2, \dots \leq \frac{2l_{\gamma_j}}{r'}.$$

Note that $\frac{1}{2}B_{j,k+1} \subset B_{j,k}$. Add all these balls to the chain $\{B'_j\}_{j=1}^M$, in between B'_j and B'_{j+1} , and renumber the sequence as B_1, B_2, \dots, B_N . Note that all the balls B_j have the same radius r' and that $\frac{1}{2}B_{j+1} \subset B_j \cap B_{j+1}$.

Next, let $A_j = \bigcup_{i=1}^j B_i$. Note that $B_0 \subset A_N$. Then $A_j \cap B_{j+1} \supset \frac{1}{2}B_{j+1}$ and the semilocal doubling property, provided by Proposition 3.8, implies that for some $\theta > 0$ (depending on $2\sigma B_0$),

$$\mu(A_j \cap B_{j+1}) \geq \mu(\tfrac{1}{2}B_{j+1}) \geq \theta \mu(B_{j+1}).$$

Since all the balls B_j have radius r' , the (q, p) -Poincaré inequality (4.1) holds for them, and we have

$$\begin{aligned} \left(\int_{B_j} |u - u_{B_j}|^q d\mu \right)^{1/q} &\leq Cr' \mu(B_j)^{1/q} \left(\int_{\lambda B_j} g_u^p d\mu \right)^{1/p} \\ &\leq C'r' \mu(2\sigma B_0)^{1/q} \left(\int_{2\sigma B_0} g_u^p d\mu \right)^{1/p} =: Q, \end{aligned}$$

where we have used the semilocal doubling property, together with the fact that $\lambda B_j \subset 2\sigma B_0$ and that r' and r_0 are fixed. Lemma 4.11 with $A = A_1$ and $E = B_2$ now yields

$$\left(\int_{A_2} |u - u_{A_2}|^q d\mu \right)^{1/q} \leq 4(1 + \theta^{-1/q})Q =: \gamma Q.$$

Another application of Lemma 4.11 with $A = A_2$ and $E = B_3$ then gives

$$\left(\int_{A_3} |u - u_{A_3}|^q d\mu \right)^{1/q} \leq \gamma^2 Q.$$

Continuing in this way along the whole sequence $\{B_j\}_{j=1}^N$, we can conclude after finitely many iterations that

$$\begin{aligned} \left(\int_B |u - u_{A_N}|^q d\mu \right)^{1/q} &\leq \left(\int_{A_N} |u - u_{A_N}|^q d\mu \right)^{1/q} \\ &\leq \gamma^N Q = C'' r' \mu(2\sigma B_0)^{1/q} \left(\int_{2\sigma B_0} g_u^p d\mu \right)^{1/p}. \end{aligned}$$

Since $r_B \geq r'$ and $\mu(B)$ and $\mu(2\sigma B_0)$ are comparable (due to the semilocal doubling property of μ), a standard argument replacing u_{A_N} by u_B in the left-hand side (at the cost of an extra factor 2 in the right-hand side, cf. [6, Lemma 4.17]) now completes the proof. \square

5. Better Poincaré inequalities

There are two types of better Poincaré inequalities. The first type of result is the Sobolev–Poincaré inequality, due to Hajlasz–Koskela [21], [22], which strengthens the left-hand side of the inequality. The arguments in [21] and [22] also show how, in sufficiently nice spaces such as \mathbf{R}^n , the dilation $\lambda > 1$ in the Poincaré inequality can be improved to $\lambda = 1$. These results were originally proved under global doubling and Poincaré assumptions, but since all the considerations in the proof are of local nature, they can also be obtained under (semi)local assumptions in the following form.

Theorem 5.1. (Local Sobolev–Poincaré inequality) *Let B_0 be a ball such that the p -Poincaré inequality (with dilation λ) and the doubling property for μ hold within B_0 in the sense of Definitions 3.1 and 4.1. Assume that the dimension condition*

$$\frac{\mu(B')}{\mu(B)} \geq C_0 \left(\frac{r_{B'}}{r_B} \right)^s \quad (5.1)$$

holds for some $C_0, s > 0$ and all balls $X \neq B' \subset B \subset B_0$.

Then there exists C , depending only on p , the doubling constant and both constants in the p -Poincaré inequality within B_0 , such that for all balls B with $2\lambda B \subset B_0$, all integrable functions u on $2\lambda B$, and all p -weak upper gradients g of u ,

$$\left(\int_B |u - u_B|^q d\mu \right)^{1/q} \leq C r_B \left(\int_{2\lambda B} g^p d\mu \right)^{1/p}, \quad (5.2)$$

where $q = p^ := sp/(s-p) > p$ if $p < s$ while $q < \infty$ is arbitrary when $p \geq s$ (in which case C depends also on q).*

If L is a local quasiconvexity constant for B_0 , in the sense that every pair of points $x, y \in B_0$ can be connected (in X) by a curve of length at most $Ld(x, y)$, then (5.2) holds for all balls B with $LB \subset B_0$, and the dilation 2λ in (5.2) can be replaced by L .

In particular, if μ is (semi)locally doubling and supports a (semi)local p -Poincaré inequality then X supports a (semi)local (p, p) -Poincaré inequality.

Proof. The first part (within B_0 and with dilation 2λ) is proved in the same way as the corresponding result with global assumptions, see e.g. Hajlasz–Koskela [21] or [22, Theorem 5.1], or [6, Theorem 4.21]. The proof of Theorem 4.39 in [6] also shows how to replace 2λ by L , since all the balls considered therein lie within $LB \subset B_0$.

To conclude the statement of the theorem under local assumptions, let $x_0 \in X$ be arbitrary and find $r_0 > 0$ so that the local assumptions hold within $B(x_0, r_0)$. Then choose a radius $0 < r'_0 \leq (5\lambda)^{-1}r_0$ so that $B'_0 := B(x_0, r'_0) \neq X$ and $\text{dist}(x_0, X \setminus B'_0) = r'_0$. For $B \subset B'_0$ it then follows that $r_B \leq 2r'_0$ and hence $2\lambda B \subset B(x_0, r_0)$. The already proved first part of the theorem then implies that (5.2) holds for B .

Under semilocal assumptions, let $B'_0 := B(x_0, r'_0)$ be arbitrary and such that $\text{dist}(x_0, X \setminus B'_0) = r'_0$. (If $B'_0 = X$, the proof is similar.) Then $r_B \leq 2r'_0$ whenever $B \subset B'_0$, and hence $2B \subset 5B'_0$. Note that the p -Poincaré inequality in the proof of [6, Theorem 4.21] is only applied to balls within $2B$, while (5.1) and the doubling property are used for balls within $2\lambda B$, where λ is the dilation in the p -Poincaré inequality within $2B$. Thus, to obtain (5.2) for $B \subset B'_0$ we need to apply the p -Poincaré inequality with λ and C_{PI} determined by $5B'_0$, followed by (5.1) and the doubling property with constants determined by $5\lambda B'_0$. This can be done because of the semilocal assumptions, since the doubling property for μ within $5\lambda B'_0$ implies (5.1) within $5\lambda B'_0$ for some $s > 0$. \square

Remark 5.2. There is a converse relation between s and q in Theorem 5.1 as well, namely if the (q, p) -Poincaré inequality (5.2) holds for all balls B with $2\lambda B \subset B_0$, then (5.1) holds with $s = qp/(q - p)$ for all balls $X \neq B' \subset B$ with $6\lambda B \subset B_0$; this follows from the proof of [6, Proposition 4.20]. Note that the formulas $s(q)$ and $q(s)$ are inverse functions of each other, if $p < s$. In particular, if we let

$$\hat{s} = \sup_{x \in X} \liminf_{r \rightarrow 0} \{s > 0 : (5.1) \text{ holds for all balls } B' \subset B(x, r)\}.$$

$$\hat{q} = \sup\{q \geq p : X \text{ supports a local } (q, p)\text{-Poincaré inequality}\},$$

then

$$\hat{q} = \begin{cases} \frac{\hat{s}p}{\hat{s} - p}, & \text{if } p < \hat{s} < \infty, \\ p, & \text{if } \hat{s} = \infty, \\ \infty, & \text{if } \hat{s} \leq p. \end{cases}$$

Note that there need not exist an optimal s (for a given B_0 in Theorem 5.1), i.e. the set of values of s for which (5.1) holds may be an open interval, cf. Example 3.1 in Björn–Björn–Lehrbäck [8]. Similarly there need not be an optimal q .

The second type of self-improvement for Poincaré inequalities is the open-ended property due to Keith–Zhong [31, Theorem 1.0.1] which strengthens the right-hand side of the inequality, see also Heinonen–Koskela–Shanmugalingam–Tyson [27, Theorem 12.3.9] and Eriksson-Bique [19]. A careful analysis of the proof (in [31] or [27]) shows that all the balls considered therein lie within a constant blow-up of the ball in the Poincaré inequality under consideration. This makes it possible to prove the following local version.

Theorem 5.3. *Assume that $p > 1$ and let $B_0 = B(x_0, r_0)$ be a ball such that $\overline{B_0}$ is compact and the p -Poincaré inequality and the doubling property for μ hold within B_0 in the sense of Definitions 3.1 and 4.1.*

Then there exist constants C , λ and $q < p$, depending only on p , the doubling constant and both constants in the p -Poincaré inequality within B_0 , such that for all balls B with $3\lambda B \subset B_0$, all integrable functions u on λB , and all q -weak upper gradients g of u ,

$$\int_B |u - u_B| d\mu \leq Cr_B \left(\int_{\lambda B} g^q d\mu \right)^{1/q}. \quad (5.3)$$

In the proof below, we will use the *inner metric* which is defined by

$$d_{\text{in}}(x, y) = \inf \text{length}(\gamma),$$

where the infimum is taken over all curves γ connecting x and y . If there are no such curves then $d_{\text{in}}(x, y) = \infty$. As X may be disconnected, this is not always a metric, but we will nevertheless still use the name “inner metric”. Balls with respect to d_{in} , defined in the obvious way, will be denoted by B_{in} .

Proof. Let λ' be the dilation constant in the p -Poincaré inequality within B_0 .

If $B_0 = X$ then the conclusion follows from Keith–Zhong [31, Theorem 1.0.1] (or [27, Theorem 12.3.9]). So without loss of generality we assume that $\text{dist}(x_0, X \setminus B_0) = r_0$. It then follows that $r_B \leq 2r_0$ for any ball $B \subset B_0$.

Lemma 4.9 shows that there exist $\Lambda \geq 1$ and $L = 24\Lambda$ such that

$$d(x, y) \leq d_{\text{in}}(x, y) \leq Ld(x, y)$$

whenever $B(x, 2\Lambda d(x, y)) \subset B_0$. It follows that if $2\Lambda B(x, r) \subset B_0$, then

$$B(x, r/L) \subset B_{\text{in}}(x, r) \subset B(x, r) \subset B_{\text{in}}(x, Lr) \subset B(x, Lr). \quad (5.4)$$

We will now explain how the arguments in the proof of [6, Theorem 4.39] (with d , L and λ therein replaced by d_{in} , 1 and $\lambda'L$) can be used to show that for every inner ball $B_{\text{in}} = B_{\text{in}}(x, r)$ such that $B(x, r) \subset B_0$, the following inner p -Poincaré inequality with dilation 1 holds:

$$\int_{B_{\text{in}}} |u - u_{B_{\text{in}}}| d\mu \leq Cr \left(\int_{B_{\text{in}}} g^p d\mu \right)^{1/p}. \quad (5.5)$$

(Since we only assume a local p -Poincaré inequality with respect to ordinary balls, Theorem 4.39 in [6] cannot be applied directly and care has to be taken when comparing ordinary and inner balls.)

More precisely, let $\rho_0 = r/2\lambda'L$ and $\rho_i = 2^{-i}\rho_0$, $i = 1, 2, \dots$. Then, as in the proof of [6, Theorem 4.39], every $x' \in B_{\text{in}}$ can be connected to x by a chain of inner balls $B_{\text{in}}^{i,j} = B_{\text{in}}(x_{i,j}, \rho_i)$ with substantial overlaps so that $d_{\text{in}}(x, x_{i,j}) + \lambda'L\rho_i < r$. (They are constructed along a geodesic from x to x' so that for each i there are boundedly many balls $B_{\text{in}}^{i,j}$, and for large i only one such ball which is, moreover, centred at x' .) Note that since $L = 24\Lambda$, we then have for $B^{i,j} = B(x_{i,j}, \rho_i)$ that

$$2\lambda'\Lambda B^{i,j} \subset \lambda'LB^{i,j} \subset B(x, r) \subset B_0.$$

Hence, for each of these balls, we have by (5.4) and the p -Poincaré inequality within B_0 that

$$\begin{aligned} \int_{B_{\text{in}}^{i,j}} |u - u_{B_{\text{in}}^{i,j}}| d\mu &\leq 2 \int_{B_{\text{in}}^{i,j}} |u - u_{B^{i,j}}| d\mu \leq C \int_{B^{i,j}} |u - u_{B^{i,j}}| d\mu \\ &\leq C' \rho_i \left(\int_{\lambda'B^{i,j}} g^p d\mu \right)^{1/p} \leq C'' \rho_i \left(\int_{\lambda'LB_{\text{in}}^{i,j}} g^p d\mu \right)^{1/p}, \end{aligned}$$

where in the last inequality it is used that $\lambda'LB^{i,j} \subset B_0$, and thus the measures

$$\mu(\lambda'B^{i,j}) \leq \mu(\lambda'LB_{\text{in}}^{i,j}) \leq \mu(\lambda'LB^{i,j})$$

are comparable by the doubling property within B_0 . The inner balls $B_{\text{in}}^{i,j}$ are then used in the standard telescopic argument proving (5.5), cf. [6, Theorem 4.39].

Next, still with respect to d_{in} and within B_0 , the proof in [27, Theorem 12.3.9] (or Keith–Zhong [31]), which is written for geodesic spaces, can be applied to show that there exists $q < p$ such that for every inner ball $B_{\text{in}} = B_{\text{in}}(x, r)$ with $768B(x, r) \subset B_0$,

$$\int_{B_{\text{in}}} |u - u_{B_{\text{in}}}| d\mu \leq C' r \left(\int_{256B_{\text{in}}} g^q d\mu \right)^{1/q}. \quad (5.6)$$

(Here it is also used that if $B_{\text{in}}(x', r') \subset 256B_{\text{in}}$ then $B_{\text{in}}(x', r') = B_{\text{in}}(x', r'')$ for some $r'' \leq 512r$, and hence

$$B(x', r'') \subset 768B(x, r) \subset B_0,$$

so (5.5) holds for every such inner ball $B_{\text{in}}(x', r') \subset 256B_{\text{in}}$ and can be used in the arguments leading to [27, Theorem 12.3.9].) Now, by (5.4),

$$B(x, r/L) \subset B_{\text{in}} \quad \text{and} \quad 256B_{\text{in}} \subset 256B(x, r),$$

all with comparable measures (depending on L). Hence, (5.6) yields (5.3) with $\lambda = 256L$ and B replaced by $B(x, r/L)$. \square

In Heinonen–Koskela–Shanmugalingam–Tyson [27, Proposition 12.3.10] it is explained how (under global assumptions) the properness of X in Keith–Zhong [31] can be relaxed to local compactness, at the price that the resulting q -Poincaré inequality only holds for $u \in N^{1,p}(\lambda B)$, which is usually enough for many applications. A counterexample by Koskela [37] shows that one cannot deduce a standard q -Poincaré inequality in this case. A similar improvement can be proved under local assumptions and in this case we *do* conclude a standard local q -Poincaré inequality, even though q may vary from ball to ball. Under local uniform assumptions there is even a fixed $q < p$, see Theorem 1.4 whose proof is given in Section 6 below.

Theorem 5.4. *If X is locally compact and supports a local p -Poincaré inequality, $p > 1$, and μ is locally doubling, then for every $x_0 \in X$ there is a ball $B'_0 \ni x_0$, and $q < p$, such that a q -Poincaré inequality holds within B'_0 in the sense of Definition 4.1.*

If X is, in addition, proper and connected, then the conclusion is semilocal, i.e. it holds for all balls $B'_0 \subset X$.

Note that for a semilocal conclusion it is *not* enough to assume that X is locally compact and that the doubling and Poincaré assumptions are semilocal. This is another consequence of the counterexample in Koskela [37].

Proof. Let $x_0 \in X$ be arbitrary and find $r_0 > 0$ so that $\overline{B(x_0, r_0)}$ is compact and the local assumptions hold within $B(x_0, r_0)$. Let λ be as given by Theorem 5.3. Then choose a radius $0 < r'_0 \leq (7\lambda)^{-1}r_0$ so that $B'_0 := B(x_0, r'_0) \neq X$ and $\text{dist}(x_0, X \setminus B'_0) = r'_0$. For $B \subset B'_0$ it then follows that $r_B \leq 2r'_0$ and hence $3\lambda B \subset B(x_0, r_0)$. The first statement then follows from Theorem 5.3.

If X is in addition proper and connected, then let $B'_0 := B(x_0, r'_0)$ be arbitrary and assume that $\text{dist}(x_0, X \setminus B'_0) = r'_0$ (the proof is similar for $B'_0 = X$). Since λ in Theorem 5.3 depends on B_0 , we cannot directly obtain semilocal conclusion from it.

Instead, let L be provided by Lemma 4.9 with B_0 replaced by $5B'_0$. Then (5.4) holds for all balls $B(x, r) \subset B'_0$, since $r \leq 2r'_0$ and $B(x, 2r) \subset 5B'_0$ hold for such balls, while the compactness requirement for $\overline{\lambda B_0}$ in Lemma 4.9 is always satisfied by the properness of X . By Proposition 3.8 and Theorem 4.4, the p -Poincaré inequality and the doubling property for μ hold within every ball in X .

Now, if $B = B(x, r) \subset B'_0$ then $768LB \subset 1537LB'_0 =: B_0$. Because the p -Poincaré inequality and the doubling property for μ hold within B_0 , we can apply

the proof of Theorem 5.3 to the ball $B_{\text{in}}(x, Lr)$ in place of B_{in} (with the doubling, Poincaré and quasiconvexity constants dictated by B_0 , while keeping L from $5B'_0$ in (5.4)). Thus, we obtain (5.6) with B_{in} replaced by $B_{\text{in}}(x, Lr)$ (and $q < p$ depending on the constants within B_0). Since

$$B(x, r) \subset B_{\text{in}}(x, Lr) \text{ (by (5.4)) and } B_{\text{in}}(x, 256Lr) \subset B(x, 256Lr),$$

all with comparable measures (by the semilocal doubling property of μ), we conclude that (5.3) holds for every $B \subset B'_0$. \square

6. Local uniform assumptions

A possible strengthening of our local assumptions is to also require uniformity in the constants. Such local uniform assumptions were used by Holopainen–Shanmugalingam [28].

Definition 6.1. The measure μ is *locally uniformly doubling* if there is a (uniform) constant C such that for each $x \in X$ there is $r > 0$ so that $\mu(2B) \leq C\mu(B)$ for all balls $B \subset B(x, r)$.

Note that there is no uniformity of the radii. Local uniform Poincaré inequalities are defined similarly. These assumptions are inherited by open subsets, and in particular they are satisfied on open subsets of spaces supporting global assumptions. Moreover, any strictly positive continuous weight on \mathbf{R}^n is locally uniformly doubling and supports a local uniform 1-Poincaré inequality. On the other hand, our local assumptions are more general, as seen in Example 6.3 below, and sufficient for many purposes, as demonstrated in this paper. However, under local uniform assumptions the constants and exponents in the local (but not necessarily the semilocal) results in Section 5 are also uniform. We now make this more precise.

Theorem 6.2. *If μ is locally uniformly doubling and X supports a local (resp. local uniform) p -Poincaré inequality, then there is $q > p$ such that X supports a local (resp. local uniform) (q, p) -Poincaré inequality.*

This is a direct consequence of Theorem 5.1, since the exponent q , given by an explicit formula, only depends on the local doubling constant.

Proof of Theorem 1.4. This follows from Theorem 5.4, since the improvement $p - q$ in the exponent depends only on the local doubling constant and the constants p , C and λ in the local Poincaré inequality within B'_0 . \square

Another consequence of local uniform assumptions is obvious in Lemma 4.9: the constants Λ and L therein are uniform. It would be interesting to know for which other results it is essential to require local uniform assumptions, especially when the consequences for the conclusions are not just the uniformity of constants.

Example 6.3. For $k \geq 3$, let

$$E_k = \bigcup_{j=2}^{\infty} ([k - 2k^{-j}, k - k^{-j}] \cup [k + k^{-j}, k + 2k^{-j}]) \times [0, k^{1-j}]$$

$$X = (\mathbf{R} \times (-\infty, 0]) \cup \bigcup_{k=3}^{\infty} E_k,$$

i.e. X is the lower half-plane together with countably many “skyscraper cities” E_k near each point $x_k = (k, 0)$, $k \geq 3$. Note that each E_k is self-similar with the factor $1/k$ and centre x_k .

Equip X with the Euclidean distance and the 2-dimensional Lebesgue measure μ . Then every $x \in X$ has a neighbourhood V_x (with respect to X) whose interior (with respect to \mathbf{R}^2) is a uniform subdomain of \mathbf{R}^2 . This implies that μ is doubling and supports a 1-Poincaré inequality on \overline{V}_x , by Theorem 4.4 in Björn–Shanmugalingam [15] and Proposition 7.1 in Aikawa–Shanmugalingam [2]. Thus μ is locally doubling and supports a local 1-Poincaré inequality.

For $j, k \geq 4$, let $r_{j,k} = k^{-j}$ and $x_{j,k} = (k + r_{j,k}, kr_{j,k}) \in E_k$. Then it is easily seen that $\mu(B(x_{j,k}, kr_{j,k}))$ is comparable to k^{1-2j} , while $\mu(B(x_{j,k}, 2kr_{j,k}))$ is comparable to $k^{2(1-j)}$. It follows that the local doubling constant near x_k is at least comparable to k .

Similarly, the ball $B(x_{j,k}, 3r_{j,k})$ is disconnected and contains the point $(k - r_{j,k}, kr_{j,k})$. Lemma 4.10 then implies that the local dilation λ_k in a Poincaré inequality around the point x_k satisfies $\lambda_k \geq \frac{1}{3}k$.

Letting $k \rightarrow \infty$ shows that μ is not locally uniformly doubling and does not support any local uniform Poincaré inequality. On the other hand, since X is proper and connected, it follows from Proposition 3.8 and Theorem 4.4 that μ is semilocally doubling and supports a semilocal 1-Poincaré inequality.

If one is satisfied with a (semi)local p -Poincaré inequality only for $p > 2$, rather than a 1-Poincaré inequality, then a simpler example can be constructed by replacing each “city” E_k with a wedge

$$V_k = \{(x, y) \in \mathbf{R}^2 : k^2|x - k| \leq y \leq 2k^2|x - k| \leq 1\}.$$

In this case, the validity of a local p -Poincaré inequality for $p > 2$ follows from [6, Example A.23].

It is also possible to make X bounded if $\mathbf{R} \times (-\infty, 0]$ in its construction is replaced by $(-1, 1) \times (-1, 0]$ and the “cities” or “wedges” are attached at the points $(1 - 8k^{-1}, 0)$ rather than at x_k , $k \geq 8$. In this case, X is not complete and does not support semilocal assumptions, because they would automatically imply global assumptions as X is bounded, and thus local uniform assumptions would follow as well.

7. Lebesgue points

Using the better Poincaré inequalities of Section 5 it can be shown that Newtonian functions have L^p -Lebesgue points q.e. also under local assumptions. By Hölder’s inequality every L^p -Lebesgue point is an L^r -Lebesgue point for all $1 \leq r \leq p$.

Theorem 7.1. *Assume that $p > 1$, that μ is locally doubling and that one of the following conditions is satisfied:*

- (a) *X is locally compact and supports a local p -Poincaré inequality;*
- (b) *for every $x \in X$ there are $r > 0$ and $q < p$ such that a q -Poincaré inequality holds within $B(x, r)$ in the sense of Definition 4.1.*

If $u \in N_{\text{loc}}^{1,p}(X)$, then q.e. x is an L^p -Lebesgue point of u , i.e.

$$\lim_{r \rightarrow 0} \oint_{B(x,r)} |u - u(x)|^p d\mu = 0.$$

On metric spaces with global assumptions such results have been obtained by Kinnunen–Latvala [33] and Björn–Björn–Parviainen [11]. Traditionally (for Sobolev functions), as well as in [33] and [11], this is shown using the density of continuous functions. Here we offer a different approach based on the fact that Newtonian functions are more precisely defined than arbitrary a.e.-representatives.

Note that, by Theorem 8.4 below, locally Lipschitz functions are dense in $N_{\text{loc}}^{1,p}(X)$ under the assumptions in Theorem 7.1, even though we do not use this fact here.

In case (a) it also follows from Theorem 9.1 below that the Newtonian functions in $N_{\text{loc}}^{1,p}(X)$ are quasicontinuous. Even though this is not known in case (b), they are still, by their Newtonian definition, more precisely defined than arbitrary a.e.-representatives, which enables us to prove the existence of Lebesgue points q.e.

Proof. Theorem 5.4 shows that the assumptions (a) imply (b). Thus in both cases we can cover X by balls B_0 such that $u \in N^{1,p}(B_0)$ and a q -Poincaré inequality and the doubling property for μ hold within B_0 , where $q < p$ depends on B_0 . Theorem 5.1 shows that if $q < p$ is chosen close enough to p , then the Sobolev–Poincaré inequality

$$\left(\int_B |u - u_B|^p d\mu \right)^{1/p} \leq C r_B \left(\int_{\lambda B} g^q d\mu \right)^{1/q}, \quad (7.1)$$

with some $\lambda \geq 1$, holds for all balls B with $\lambda B \subset B_0$ and for every upper gradient g of u .

For $x \in B_0$, let $r_j = 2^{-j}$ and $v(x) = \limsup_{j \rightarrow \infty} v_j(x)$, where

$$v_j(x) = \left(\int_{B(x, r_j)} |u - u(x)|^p d\mu \right)^{1/p}, \quad j = 0, 1, \dots \quad (7.2)$$

Note that $u \in L^p(B_0)$ and hence, by the Lebesgue differentiation theorem (Theorem 3.9), $v = 0$ a.e. in B_0 . We shall show that $v \in N^{1,p}(B_0)$ and hence $v = 0$ q.e. in B_0 , by [6, Proposition 1.59].

To this end, let $\gamma: [0, l_\gamma] \rightarrow B_0$ be a nonconstant rectifiable curve (parameterized by arc length) and $g \in L^p(B_0)$ be an upper gradient of u . By splitting γ into parts, if necessary, and by considering sufficiently large j , we can assume that $\frac{1}{2}r_j \leq l_\gamma \leq r_j$ and that $\frac{5}{2}\lambda B \subset B_0$, where $B := B(x, 2r_j)$ with $x = \gamma(0)$ and $y = \gamma(l_\gamma)$ being the endpoints of γ . Since $u \in L^p(B_0)$, both $v_j(x)$ and $v_j(y)$ are finite, and hence

$$\begin{aligned} |v_j(x) - v_j(y)| &\leq |v_j(x) - |u_B - u(x)|| + |v_j(y) - |u_B - u(y)|| + |u(x) - u(y)| \\ &\leq \left(\int_{B(x, r_j)} |u - u_B|^p d\mu \right)^{1/p} + \left(\int_{B(y, r_j)} |u - u_B|^p d\mu \right)^{1/p} + \int_\gamma g ds, \end{aligned}$$

where we have used that by the triangle inequality for the normalized L^p -norm,

$$\begin{aligned} |v_j(x) - |u_B - u(x)|| &= \left| \left(\int_{B(x, r_j)} |u - u(x)|^p d\mu \right)^{1/p} - \left(\int_{B(x, r_j)} |u_B - u(x)|^p d\mu \right)^{1/p} \right| \\ &\leq \left(\int_{B(x, r_j)} |(u - u(x)) - (u_B - u(x))|^p d\mu \right)^{1/p}, \end{aligned}$$

and similarly for $|v_j(y) - |u_B - u(y)||$.

The local doubling property within B_0 and (7.1) then imply that

$$\begin{aligned} |v_j(x) - v_j(y)| &\leq C' \left(\int_B |u - u_B|^p d\mu \right)^{1/p} + \int_\gamma g ds \\ &\leq C'' r_j \left(\int_{\lambda B} g^q d\mu \right)^{1/q} + \int_\gamma g ds \\ &\leq C'' r_j \inf_{z \in B} g_M(z) + \int_\gamma g ds \\ &\leq C''' \int_\gamma (g_M + g) ds, \end{aligned} \quad (7.3)$$

where C''' is independent of j and

$$g_M^q(z) := \sup_{\substack{B' \ni z \\ \frac{5}{2}B' \subset B_0}} \int_{B'} g^q d\mu \quad (7.4)$$

is the local noncentred maximal function of g^q within B_0 . Glueing together all the parts of γ and by applying [6, Lemma 1.52] we conclude that $C'''(g_M + g)$ is a p -weak upper gradient of v in B_0 . Standard arguments as in the proof of [6, Lemma 3.12 and Theorem 3.13] (and the observation that $\mu(5B') \leq C_1\mu(B')$ for all balls B' appearing in the definition of g_M) imply that the maximal function is bounded on $L^{p/q}(B_0)$ which yields that

$$\int_{B_0} g_M^p d\mu \leq C_0 \int_{B_0} g^p d\mu,$$

and hence $v \in N^{1,p}(B_0)$, as required.

As X is Lindelöf there is a countable cover of X by balls of the type above, and since the capacity is countably subadditive we conclude the existence of Lebesgue points q.e. \square

Remark 7.2. Since Lebesgue points are of a local nature, the proof of Theorem 7.1 can be modified so that

$$\lim_{r \rightarrow 0} \int_{B(x,r)} |u - u(x)|^{q(x)} d\mu = 0 \quad (7.5)$$

holds for q.e. x and all

$$q(x) < \begin{cases} \frac{s(x)p}{s(x) - p}, & \text{if } s(x) > p, \\ \infty, & \text{if } s(x) \leq p, \end{cases}$$

where

$$s(x) = \liminf_{r \rightarrow 0} \{s > 0 : (5.1) \text{ holds for all balls } B' \subset B \subset B(x, r)\}.$$

If μ is locally uniformly doubling, then $\hat{s} := \sup_x s(x) < \infty$, and we can use \hat{s} instead of $s(x)$ to find a common $q = q(x) > p$ so that (7.5) holds for q.e. x .

In fact, if $s(x)$ is “attained” then it is even possible to reach the borderline case, as in Heinonen–Koskela–Shanmugalingam–Tyson [27, Theorem 9.2.8]. More precisely, if there is $r_0 > 0$ such that

$$(5.1) \text{ holds for all balls } B' \subset B \subset B(x, r_0) \text{ with } s = s(x) > p,$$

then we may let $q(x) = s(x)p/(s(x) - p)$. To see this, one uses the following estimate with $q = q(x)$ and $y \in B(x, r_0)$,

$$\begin{aligned} & \limsup_{r \rightarrow 0} \left(\int_{B(y,r)} |u - u(y)|^q d\mu \right)^{1/q} \\ & \leq \limsup_{r \rightarrow 0} \left(\int_{B(y,r)} |u - u_{B(y,r)}|^q d\mu \right)^{1/q} + \limsup_{r \rightarrow 0} |u_{B(y,r)} - u(y)| \\ & \leq C \limsup_{r \rightarrow 0} \left(r^p \int_{B(y,r)} g_u^p d\mu \right)^{1/p} + \limsup_{r \rightarrow 0} \int_{B(y,r)} |u - u(y)| d\mu, \end{aligned}$$

where the (q, p) -Poincaré inequality within $B(x, r_0)$ is provided by Theorem 5.1. The second term tends to 0 q.e., as we already know that u has L^1 -Lebesgue points q.e., while the first term tends to 0 q.e. in $B(x, r_0)$ by Lemma 9.2.4 in Heinonen–Koskela–Shanmugalingam–Tyson [27].

In particular, if for each $x \in X$ there is $r > 0$ such that (5.1) holds for all balls $B' \subset B \subset B(x, r)$ with the same $s > 0$ (independent of x), then u has L^q -Lebesgue points q.e., with $q = sp/(s - p)$ for $p < s$ and all $q < \infty$ for $p \geq s$, provided that the assumptions in Theorem 7.1 are fulfilled.

Remark 7.3. The proof of Theorem 7.1 shows that the assumptions can be further relaxed with a somewhat weaker conclusion. Namely, if μ is locally doubling and only supports a local p -Poincaré inequality (which need not improve to a q -Poincaré inequality as X is not necessarily locally compact), then it can be verified that the function v , defined by (7.2), belongs to $N^{1,q}(B_0)$ for every $1 \leq q < p$. To see this one replaces q by p in (7.3) and (7.4), and then uses the L^1 to weak- L^1 boundedness of the maximal function, which yields that g_M belongs to weak- $L^p(B_0)$ and thus to $L^q(B_0)$ for all $q < p$.

An immediate consequence is that $v = 0$ outside a set of zero q -capacity and hence, every $u \in N_{\text{loc}}^{1,p}(X)$ has Lebesgue points q -quasieverywhere. Since it is not clear whether μ supports a local q -Poincaré inequality for some $q < p$, this cannot be deduced directly from the inclusion $N_{\text{loc}}^{1,p}(X) \subset N_{\text{loc}}^{1,q}(X)$ as we have no q -quasieverywhere Lebesgue point result available for functions in $N_{\text{loc}}^{1,q}(X)$ under these assumptions.

8. Density of Lipschitz functions

Density of smooth functions is a useful property of Sobolev spaces, with many important consequences (which we will discuss in Section 9). In metric spaces the smoothest functions to consider are (locally) Lipschitz functions. There are two types of density results for $N^{1,p}(X)$ in the literature. The first one is due to Shanmugalingam [40] (it can also be found in [6, Theorem 5.1] and [27, Theorem 8.2.1]).

Theorem 8.1. (Shanmugalingam [40, Theorem 4.1]) *If μ is globally doubling and supports a global p -Poincaré inequality, then Lipschitz functions are dense in $N^{1,p}(X)$.*

The following result was recently obtained by Ambrosio–Colombo–Di Marino [3] and Ambrosio–Gigli–Savaré [4]. In fact, it is not explicitly spelt out in either paper, but it is a direct consequence of a combination of results in the two papers. Below we explain this in some detail, see Remark 8.9. Note that by density we always mean norm-density in the $N^{1,p}$ norm, with the exception of the discussion in Remark 8.9.

Theorem 8.2. (Ambrosio et al. [3], [4]) *Assume that X is a complete globally doubling metric space and that $p > 1$. Then Lipschitz functions are dense in $N^{1,p}(X)$.*

The main difference in these two results is that the former assumes doubling for μ (and not just for X) together with a Poincaré inequality, while the latter requires completeness and $p > 1$. Note that even though the doubling property and the Poincaré inequality extend from X to its closure \bar{X} , it need not be true that $N^{1,p}(\bar{X}) = N^{1,p}(X)$, cf. [27, Lemma 8.2.3]. In other words, completeness (or at least local compactness) is not a negligible assumption. Thus both results have their pros and cons. Our aim in this section is to extend both of these results to local assumptions and to combine them into a unified result (when $p > 1$).

Remark 8.3. Without both completeness and a global Poincaré inequality, Lipschitz functions are not necessarily dense in $N^{1,p}(X)$, consider e.g. $X = \mathbf{R} \setminus \{0\}$ or

the slit disc in \mathbf{R}^2 , both of which support a local 1-Poincaré inequality. This also shows that the completeness assumption in Theorem 8.2 cannot be dropped nor replaced by local compactness.

It is therefore natural to obtain density of locally Lipschitz functions in most of our results below. It should be mentioned that there is no known example of a metric space X such that locally Lipschitz functions are not dense in $N^{1,p}(X)$. (A function $u : X \rightarrow \mathbf{R}$ is *locally Lipschitz* if for every $x \in X$ there is $r > 0$ such that $u|_{B(x,r)}$ is Lipschitz.)

The following is a local generalization of Theorem 8.1.

Theorem 8.4. *If μ is locally doubling and supports a local p -Poincaré inequality, then locally Lipschitz functions are dense in $N_{\text{loc}}^{1,p}(X)$.*

Note that if Ω is an open subset of X , then the local assumptions are inherited by Ω and hence we can also directly conclude the density of locally Lipschitz functions in $N_{\text{loc}}^{1,p}(\Omega)$. The same is true for Theorem 8.6 below.

To prove Theorem 8.4 we will need the following lemma.

Lemma 8.5. *Assume that the p -Poincaré inequality and the doubling property for μ hold within a ball $2B_0$ in the sense of Definitions 3.1 and 4.1. Then every $u \in N^{1,p}(2B_0)$ can be approximated in the $N^{1,p}(B_0)$ -norm by Lipschitz functions u_k with support in $2B_0$. Moreover, $\mu(\{x \in B_0 : u(x) \neq u_k(x)\}) \rightarrow 0$ as $k \rightarrow \infty$.*

Proof. By Proposition 3.4, B_0 can be covered by finitely many balls B_j with centres in B_0 and radii $r' = r_{B_0}/22\lambda$, where λ is the dilation constant in the p -Poincaré inequality within $2B_0$. Let $\{\varphi_j\}_j$ be a Lipschitz partition of unity on $\bigcup_j B_j$ subordinate to $2B_j$, e.g. one constructed as in the proof of Theorem 8.4 below.

Let $u \in N^{1,p}(2B_0)$ be arbitrary and let $g \in L^p(2B_0)$ be an upper gradient of u . Noting that $\max\{\min\{u, k\}, -k\} \rightarrow u$ in $N^{1,p}(2B_0)$, as $k \rightarrow \infty$, we can assume without loss of generality that $|u| \leq 1$. For $x \in 2B_0$ define the local noncentred maximal function

$$g_M^p(x) = \sup_{\substack{B \ni x \\ \frac{5}{2}B \subset 2B_0}} \int_B g^p d\mu \quad \text{and} \quad E_t = \{x \in 2B_0 : g_M(x) > t\}.$$

Standard arguments as in the proof of [6, Lemma 3.12] (and the observation that $\mu(5B) \leq C_1\mu(B)$ for all balls B appearing in the definition of g_M) then imply that

$$t^p \mu(E_t) \leq C_2 \int_{E_t} g^p d\mu \rightarrow 0, \quad \text{as } t \rightarrow \infty. \quad (8.1)$$

For a fixed j and $x \in 2B_j \setminus E_t$ we have as in the proof of [6, Theorem 5.1] or Shanmugalingam [40, Theorem 4.1] that for all $0 < \frac{1}{2}r \leq \rho \leq r \leq 8r'$,

$$|u_{B(x,\rho)} - u_{B(x,r)}| \leq C_3 r \left(\int_{\lambda B(x,r)} g^p d\mu \right)^{1/p} \leq C_3 r g_M(x) \leq C_3 t r,$$

since for such radii, $\frac{5}{2}\lambda B(x,r) \subset 22\lambda B_j \subset 2B_0$. A telescopic argument as in the proof of [6, Theorem 5.1] then shows that the limit $\bar{u}(x) = \lim_{r \rightarrow 0} u_{B(x,r)}$ exists everywhere in $2B_j \setminus E_t$ and is $C_4 t$ -Lipschitz therein. Also, by Lebesgue's differentiation theorem (Theorem 3.9), $\bar{u} = u$ a.e. in $2B_j \setminus E_t$. Using e.g. truncated McShane extensions, \bar{u} extends to a $C_4 t$ -Lipschitz function $u_{t,j}$ on $2B_j$ such that $|u_{t,j}| \leq 1$. Then also $u_t = \sum_j \varphi_j u_{t,j}$ equals u a.e. in $B_0 \setminus E_t$. It follows that $(g + C_4 t)\chi_{E_t \cap B_0}$ is a p -weak upper gradient of $u - u_t$ and hence

$$\|u - u_t\|_{N^{1,p}(B_0)}^p \leq 2^p \mu(E_t \cap B_0) + (\|g\|_{L^p(E_t \cap B_0)} + C_4 t \mu(E_t \cap B_0)^{1/p})^p.$$

Since $g \in L^p(2B_0)$, we conclude from (8.1) that $u_t \rightarrow u$ in $N^{1,p}(B_0)$. By construction, u_t is Lipschitz in $2B_0$ and $\text{supp } u_t \subset \bigcup_j 2B_j \subset 2B_0$. \square

Proof of Theorem 8.4. Let $u \in N_{\text{loc}}^{1,p}(X)$. For every $x \in X$, let $B_x = B(x, r_x)$ be a ball such that $u \in N^{1,p}(B_x)$ and such that the p -Poincaré inequality and the doubling property for μ hold within B_x . As X is Lindelöf, we can find a countable subcollection such that $X = \bigcup_{j=1}^{\infty} \frac{1}{4}B_{x_j}$. Let $B_j = \frac{1}{4}B_{x_j}$, $j = 1, 2, \dots$

We construct a suitable Lipschitz partition of unity. For each j we find $\psi_j \in \text{Lip}(X)$ such that $\chi_{B_j} \leq \psi_j \leq \chi_{2B_j}$. Let inductively $\varphi_1 = \psi_1$ and

$$\varphi_j := \psi_j \left(1 - \sum_{k=1}^{j-1} \varphi_k \right), \quad j \geq 2.$$

Then $\sum_{k=1}^j \varphi_k = 1$ in B_j , and hence $\varphi_k \equiv 0$ in B_j for $k > j$, so that

$$\sum_{k=1}^{\infty} \varphi_k = 1 \quad \text{in } B_j.$$

As this holds for all B_j we see that $\{\varphi_j\}_{j=1}^{\infty}$ is a partition of unity.

Since each $4B_j = B_{x_j}$, the assumptions of Lemma 8.5 are satisfied for each $2B_j$ (in place of B_0). Hence, for every $\varepsilon > 0$ and each j , there is $v_j \in \text{Lip}(2B_j)$ with

$$\|u - v_j\|_{N^{1,p}(2B_j)} \leq \frac{2^{-j}\varepsilon}{1 + L_j},$$

where L_j is the Lipschitz constant of φ_j . Then also

$$\begin{aligned} \|\varphi_j(u - v_j)\|_{N^{1,p}(2B_j)}^p &\leq \|u - v_j\|_{L^p(2B_j)}^p + (L_j \|u - v_j\|_{L^p(2B_j)} + \|g_{u-v_j}\|_{L^p(2B_j)})^p \\ &\leq 2(1 + L_j)^p \|u - v_j\|_{N^{1,p}(2B_j)}^p \leq 2^{1-pj} \varepsilon^p. \end{aligned}$$

As $v := \sum_{j=1}^{\infty} \varphi_j v_j$ is a locally finite sum of Lipschitz functions, v is locally Lipschitz. Combining this with the above estimate we conclude that

$$\|u - v\|_{N^{1,p}(X)} \leq \sum_{j=1}^{\infty} \|\varphi_j(u - v_j)\|_{N^{1,p}(2B_j)} \leq 2^{1/p} \varepsilon. \quad \square$$

We now turn to Theorem 8.2 which we generalize in the following way, also taking into account Theorem 8.1. Note that the set of points, for which (a) resp. (b) below holds, is open. Thus, if X is connected, these two sets cannot be disjoint.

Theorem 8.6. *Let $p > 1$ and assume that for every $x \in X$ there is a ball $B_x \ni x$ such that either*

- (a) B_x is locally compact and globally doubling; or
- (b) the p -Poincaré inequality and the doubling property for μ hold within B_x in the sense of Definitions 3.1 and 4.1.

Then locally Lipschitz functions are dense in $N_{\text{loc}}^{1,p}(X)$.

In particular, this holds if $p > 1$ and X is locally compact and locally doubling.

Proof. Without loss of generality we can replace the assumption (a) by

- (a') $\overline{B_x}$ is compact and globally doubling.

The proof is now the same as the proof of Theorem 8.4, but with appeal to Theorem 8.2 instead of Lemma 8.5 for the balls B_x satisfying (a'). \square

So far we have deduced the density for locally Lipschitz functions. A natural question is when density can be obtained for Lipschitz functions. Note that under the assumptions in Theorems 8.4 and 8.6 it can happen that Lipschitz functions are not dense, see Remark 8.3.

Theorem 8.7. *Assume that μ is semilocally doubling and supports a semilocal p -Poincaré inequality. Then Lipschitz functions with bounded support are dense in $N^{1,p}(X)$. If X is also proper, then $\overline{\text{Lip}_c(X)} = N^{1,p}(X)$.*

$\text{Lip}_c(X)$ denotes the space of Lipschitz functions with compact support.

Proof. Let $u \in N^{1,p}(X)$ and $\varepsilon > 0$. Find a sufficiently large ball B with $r_B > 1$ such that $\|u\|_{N^{1,p}(X \setminus \overline{B})} < \varepsilon$. By Lemma 8.5 and the semilocal assumptions, there is $v \in \text{Lip}(X)$ so that $\|u - v\|_{N^{1,p}(2B)} < \varepsilon$.

For $\eta(x) = (1 - \text{dist}(x, B))_+$ we then get that $v_\varepsilon := v\eta \in \text{Lip}(X)$ with $\text{supp } v_\varepsilon \subset 2B$ and $u - v_\varepsilon = u(1 - \eta) + (u - v)\eta$. Since

$$g_{u(1-\eta)} \leq |u|g_\eta + (1 - \eta)g_u \quad \text{and} \quad g_{(u-v)\eta} \leq \eta g_{u-v} + |u - v|g_\eta,$$

a simple calculation yields $\|u - v_\varepsilon\|_{N^{1,p}(X)} < 6\varepsilon$, which concludes the proof of the first part. If X is also proper then v_ε has compact support and thus $\overline{\text{Lip}_c(X)} = N^{1,p}(X)$. \square

Proof of Theorem 1.5. The assumptions in Theorem 8.7 are satisfied because of Proposition 3.8 and Theorem 4.4, and hence the result follows. \square

Theorem 8.8. *Assume that X is a complete semilocally doubling metric space and that $p > 1$. Then $\overline{\text{Lip}_c(X)} = N^{1,p}(X)$.*

Note that, by Lemma 3.6 and Proposition 3.8, a metric space is complete and semilocally doubling if and only if it is proper and locally doubling. A natural question is if Theorem 8.8 holds when X is only complete and locally doubling. The completeness in Theorem 8.8 cannot be replaced by local compactness, see Remark 8.3.

Proof. This is quite similar to the proof of Theorem 8.7. Let $u \in N^{1,p}(X)$ and $\varepsilon > 0$. Find a sufficiently large ball B with $r_B > 1$ such that $\|u\|_{N^{1,p}(X \setminus \overline{B})} < \varepsilon$. By the semilocal assumptions, $\overline{2B}$ is a complete globally doubling metric space. Hence by Theorem 8.2 there is $v \in \text{Lip}(\overline{2B})$ so that $\|u - v\|_{N^{1,p}(2B)} < \varepsilon$. The rest of the proof is identical to the second half of the proof of Theorem 8.7, since X is proper by Lemma 3.6. \square

Remark 8.9. We will now explain how Theorem 8.2 follows from Ambrosio–Colombo–Di Marino [3] and Ambrosio–Gigli–Savaré [4]. In [4], a function g is called a p -upper gradient of a function f if there is \tilde{f} such that $\tilde{f} = f$ a.e. and g is a p -weak upper gradient of f in the sense of Definition 2.1. In [4] they also define p -weak upper gradients (different from ours), p -relaxed upper gradients and p -relaxed slopes. Furthermore, they show that if a function $f \in L^p(X)$ has a gradient $g \in L^p(X)$ in any of these four senses, then it has one in each of the four senses and the a.e.-minimal ones coincide. This is shown in [4, Theorem 7.4 and Section 8.1] assuming completeness of X .

In Ambrosio–Colombo–Di Marino [3], there is a different definition of p -relaxed slope and the same definition of p -weak upper gradient as in [4]. In [3, Theorem 6.1] it is shown that if X is complete and $f \in L^p(X)$ has a gradient $g \in L^p(X)$ in either of these two senses, then it has one in the other and the a.e.-minimal ones coincide. So in conclusion the a.e.-minimal gradients in $L^p(X)$ in all five senses

exist simultaneously and then coincide, assuming that $f \in L^p(X)$ and that X is complete. Moreover, the Sobolev space $W^{1,p}(X)$ in [3, Corollary 7.5] thus coincides with

$$\widehat{N}^{1,p}(X) = \{u : u = v \text{ a.e. for some } v \in N^{1,p}(X)\} \quad (8.2)$$

(recall that functions in $N^{1,p}(X)$ are defined pointwise everywhere).

The following density result is a special case of the equivalence result from Theorem 7.4 (and Section 8.1) in [4]. (Note that this is not a norm-density result as elsewhere in this section.)

Theorem 8.10. *Assume that X is complete and $p > 1$. Let $f \in N^{1,p}(X)$. Then there exist Lipschitz functions f_n such that*

$$\lim_{n \rightarrow \infty} \left(\int_X |f_n - f|^p d\mu + \int_X |\text{Lip } f_n - g_f|^p d\mu \right) = 0,$$

where

$$\text{Lip } f_n(x) := \limsup_{r \rightarrow 0} \sup_{y \in B(x,r)} \frac{|f_n(y) - f_n(x)|}{r}$$

is the local upper Lipschitz constant (also called upper pointwise dilation) of f_n , and g_f is the minimal p -weak upper gradient of f (in the sense of Definition 2.1).

As explained at the bottom of p. 1 of [3], once we know reflexivity, the norm-density of Lipschitz functions follows directly using Mazur's lemma. This reflexivity was obtained in Corollary 7.5 in [3] when X is complete and doubling, and thus Theorem 8.2 follows.

Completeness is not assumed explicitly in [3, Corollary 7.5], but the result relies on other results in [3] (e.g. Theorem 7.4) for which completeness is assumed. And indeed, the completeness assumption cannot be dropped or replaced by assuming that X is merely locally compact for the density result, since norm-density of Lipschitz functions then can fail, see Remark 8.3. The same counterexamples show that Theorem 8.10 cannot hold (in general) in locally compact spaces with Lipschitz functions f_n . Whether it can hold in locally compact spaces, if f_n are just required to be locally Lipschitz, is not clear because the “partition of unity” technique used in the proof of Theorem 8.4 cannot be used to construct such an extension, at least not in such an easy way as here, since it would require controlling $\|\text{Lip } v - g_f\|$ in terms of $\|\text{Lip}(\varphi_j v_j) - g_{f_j}\|$.

A slight word of warning: one may get the impression that as a consequence of Theorem 8.10 one can deduce that $g_f = \text{Lip } f$ a.e. for Lipschitz functions. This is not so, and indeed it is not true in general, as seen by considering e.g. the von Koch snowflake curve, on which $g_f \equiv 0$ for all functions, because of the lack of rectifiable curves. The equality $g_f = \text{Lip } f$ a.e. for Lipschitz functions is however true if X is complete and μ is globally doubling and supports a global p -Poincaré inequality, by Theorem 6.1 in Cheeger [17].

Remark 8.11. Occasionally it may be interesting to know when (locally) Lipschitz or continuous functions are dense in $N^{1,p}(X)$ even when $X \neq \text{supp } \mu$, in which case our general condition that all balls have positive measure is invalid. It turns out that this happens if and only if they are dense in $N^{1,p}(\text{supp } \mu)$. For Lipschitz and continuous functions this is Lemma 5.19 (e) in [6]. For locally Lipschitz functions this can be proved similarly, provided that one uses the locally Lipschitz extensions due to Luukkainen–Väisälä [39, Theorem 5.7]. (Note that the class LIP in [39] consists of locally Lipschitz functions.)

For quasicontinuity, which will be discussed in the next section, a similar equivalence is also true, by [6, Lemma 5.19 (d)].

9. Quasicontinuity and other consequences

Having established the density of continuous (or more exactly locally Lipschitz) functions we can now draw a number of qualitative conclusions about Newtonian functions and capacities.

Throughout this section, $\Omega \subset X$ is open. A function $u : \Omega \rightarrow \overline{\mathbf{R}}$ is *quasicontinuous* if for every $\varepsilon > 0$ there is an open set G with $C_p(G) < \varepsilon$ such that $u|_{\Omega \setminus G}$ is continuous. See the recent paper Björn–Björn–Malý [10] for several different characterizations of quasicontinuity, and in particular that one can equivalently replace the condition $C_p(G) < \varepsilon$ by $C_p^\Omega(G) < \varepsilon$, where C_p^Ω is the capacity associated with $N^{1,p}(\Omega)$ rather than with $N^{1,p}(X)$. Note also that, in the following theorem, X can be replaced by any open subset of X . Moreover, the conditions in (a) and (b) below are inherited by open subsets.

Theorem 9.1. *Assume that X is locally compact and that one of the following conditions is satisfied:*

- (a) μ is locally doubling and supports a local p -Poincaré inequality,
- (b) $p > 1$ and X is locally doubling, or more generally the conditions in Theorem 8.6 are satisfied;
- (c) continuous functions are dense in $N^{1,p}(X)$.

Then every $u \in N_{\text{loc}}^{1,p}(X)$ is quasicontinuous in X and hence C_p is an outer capacity, i.e.

$$C_p(E) = \inf_{\substack{G \supset E \\ G \text{ open}}} C_p(G) \quad \text{for every } E \subset X.$$

Quasicontinuity has been established for Newtonian functions under various assumptions in Björn–Björn–Shanmugalingam [12] and Björn–Björn–Lehrbäck [8] for open subsets. See also Shanmugalingam [40]. Assuming that X is complete and that μ is globally doubling and supports a global p -Poincaré inequality, quasicontinuity was also established for functions in $N^{1,p}(U)$ when U is a quasiopen subset of X , by Björn–Björn–Latvala [7] (when $p > 1$) and Björn–Björn–Malý [10].

For $p > 1$, quasicontinuity also implies that C_p is a Choquet capacity and thus, if X is locally compact, that all Borel sets are capacitable, i.e.

$$C_p(E) = \sup_{\substack{K \subset E \\ K \text{ compact}}} C_p(K) \quad \text{for every Borel set } E \subset X,$$

see e.g. Aikawa–Essén [1, Part 2, Section 10] together with [6, Theorems 6.4 and 6.7 (viii)]. It should be mentioned that there is no known example of a Newtonian function which is not quasicontinuous, nor of a metric space X such that continuous functions are not dense in $N^{1,p}(X)$.

Proof. By Theorems 8.4 and 8.6, (a) \Rightarrow (c) and (b) \Rightarrow (c). So assume that (c) holds. By [6, Theorem 5.21] every $u \in N_{\text{loc}}^{1,p}(X)$ has a quasicontinuous representative. As X is locally compact, Proposition 4.7 in Björn–Björn–Lehrbäck [8] then shows that every $u \in N_{\text{loc}}^{1,p}(X)$ is quasicontinuous. The outer capacity property then follows from Björn–Björn–Shanmugalingam [12, Corollary 1.3] (or [6, Theorem 5.31]). \square

Quasicontinuity of $N^{1,p}(X)$ gets inherited to open subsets in the following way. Recall the definition of $\widehat{N}^{1,p}(\Omega)$ in (8.2).

Proposition 9.2. *If every $u \in N^{1,p}(X)$ is quasicontinuous, then $N_{\text{loc}}^{1,p}(\Omega)$ consists exactly of those $u \in \widehat{N}_{\text{loc}}^{1,p}(\Omega)$ which are quasicontinuous, and similarly for $N^{1,p}(\Omega)$.*

Proof. Clearly, $N_{\text{loc}}^{1,p}(\Omega) \subset \widehat{N}_{\text{loc}}^{1,p}(\Omega)$. Multiplying $u \in N_{\text{loc}}^{1,p}(\Omega)$ by Lipschitz cut-off functions shows that for each $x \in \Omega$ there is $r_x > 0$ such that u is quasicontinuous in $B(x, r_x)$. As X is Lindelöf, a countable covering of Ω by such balls yields quasicontinuity in Ω .

Conversely, by an argument due to Kilpeläinen [32], every quasicontinuous $u \in \widehat{N}_{\text{loc}}^{1,p}(\Omega)$ is q.e. equal to a Newtonian function and hence itself in $N_{\text{loc}}^{1,p}(\Omega)$, cf. [6, Propositions 5.22 and 5.23]. \square

Quasicontinuity, or rather the outer capacity property following from it, provides us with a short proof of the following fact, cf. Kallunki–Shanmugalingam [30] where it was proved under stronger assumptions. A similar statement is not true if we drop the assumption that K be compact. (Let e.g. K be a countable dense subset of a ball in \mathbf{R}^n and $p \leq n$.)

Proposition 9.3. *Assume that C_p is an outer capacity and that continuous resp. (locally) Lipschitz functions are dense in $N^{1,p}(X)$. If $K \subset X$ is compact, then*

$$C_p(K) = \inf \|u\|_{N^{1,p}(X)}^p,$$

where the infimum is taken over all continuous resp. (locally) Lipschitz u such that $u \geq 1$ on K .

Note that if continuous functions are dense in $N^{1,p}(X)$, then the condition that C_p is an outer capacity is equivalent to requiring that all functions in $N^{1,p}(X)$ are quasicontinuous, see Theorems 5.20, 5.31 and Proposition 5.32 in [6].

Proof. Given $\varepsilon > 0$, there exist an open set $G \supset K$ and $u \in N^{1,p}(X)$ such that $u = 1$ on G and

$$\|u\|_{N^{1,p}(X)}^p < C_p(K) + \varepsilon. \quad (9.1)$$

Since K is compact, there exists $0 < \delta \leq 1$ such that $\text{dist}(K, X \setminus G) > 2\delta$. Let $\eta(x) := \min\{1, \text{dist}(x, K)/\delta\}$ and set $\tilde{v} = u + \eta(v - u)$, where v is continuous resp. (locally) Lipschitz in X and such that $\|v - u\|_{N^{1,p}(X)} < \varepsilon\delta$. Then $\tilde{v} = 1$ on K and, as $g_{\eta(v-u)} \leq \eta g_{v-u} + |v - u|g_\eta$, we also have

$$\|g_{\eta(v-u)}\|_{L^p(X)} \leq \|g_{v-u}\|_{L^p(X)} + \frac{1}{\delta} \|v - u\|_{L^p(X)} \leq \frac{2}{\delta} \|v - u\|_{N^{1,p}(X)} < 2\varepsilon.$$

It then follows that

$$\begin{aligned} C_p(K) &\leq \|\tilde{v}\|_{N^{1,p}(X)}^p \\ &\leq (\|u\|_{L^p(X)} + \|v - u\|_{L^p(X)})^p + (\|g_u\|_{L^p(X)} + \|g_{\eta(v-u)}\|_{L^p(X)})^p \\ &\leq (\|u\|_{L^p(X)} + \varepsilon\delta)^p + (\|g_u\|_{L^p(X)} + 2\varepsilon)^p, \end{aligned}$$

which, by (9.1), tends to $C_p(K)$ as $\varepsilon \rightarrow 0$.

Finally, $\tilde{v} = 1 - \eta + \eta v$ is continuous resp. (locally) Lipschitz in G (as $u \equiv 1$ there), while $\tilde{v} = v$ is continuous resp. (locally) Lipschitz in the open set

$$X \setminus \text{supp}(1 - \eta) \supset X \setminus G.$$

It follows that \tilde{v} is continuous resp. locally Lipschitz in X and, as $\text{dist}(X \setminus G, \text{supp}(1 - \eta)) > \delta$, also Lipschitz in X whenever v is Lipschitz. \square

10. Conclusions for p -harmonic functions

Nonlinear potential theory associated with p -harmonic functions, $p > 1$, and quasi-minimizers has been extensively studied during the last 20 years on complete metric spaces equipped with globally doubling measures supporting a global p -Poincaré inequality, see e.g. [6] and the references therein. It is therefore natural to see to which extent this theory can be extended to local assumptions.

In much of this theory the properness of X plays an important role and even though some of the theory has already been developed on noncomplete spaces (see in particular Kinnunen–Shanmugalingam [36], Björn [14] and Björn–Marola [13]), we will in this section restrict ourselves to *proper* X . We will also assume that X is connected, that μ is locally doubling and supports a local p -Poincaré inequality, and that $p > 1$.

As we have seen, it then follows from Proposition 3.8 and Theorem 4.4 that μ is semilocally doubling and supports a semilocal p -Poincaré inequality. The results in this paper show that most of the essential tools needed to develop the potential theory on metric spaces are available also under these assumptions.

Definition 10.1. A function $u \in N_{\text{loc}}^{1,p}(\Omega)$ is a Q -quasi(super)minimizer in Ω if

$$\int_{\varphi \neq 0} g_u^p d\mu \leq Q \int_{\varphi \neq 0} g_{u+\varphi}^p d\mu \quad (10.1)$$

for all (nonnegative) $\varphi \in \text{Lip}_c(\Omega)$.

If $Q = 1$ in (10.1) then u is a (super)minimizer. A p -harmonic function is a continuous minimizer.

See Björn [5, Proposition 3.2] for equivalent ways of defining quasisuperminimizers; those equivalences also extend to spaces with our local assumptions. (Here Theorem 8.7 is needed.) Our first observation is that interior regularity is preserved under local assumptions. A function u on Ω is *lsc-regularized* if $u(x) = \text{ess lim inf}_{y \rightarrow x} u(y)$ for all $x \in \Omega$.

Theorem 10.2. Let u be a quasi(super)minimizer in Ω . Then u has a representative \tilde{u} which is continuous (resp. lsc-regularized).

Moreover, the weak Harnack inequalities for quasi(super)minimizers hold within every ball $B_0 \subset X$ i.e. for every ball $B \subset B_0$ with $\Lambda B \subset \Omega$, where Λ and the weak Harnack constants depend only on B_0 .

See e.g. Kinnunen–Martio [34], [35], Kinnunen–Shanmugalingam [36], Björn–Marola [13] and Björn–Björn [6] for formulations of the weak Harnack inequalities. There are various types of weak Harnack inequalities in these papers and under different assumptions. In [34], [35] and [36] a q -Poincaré inequality for some $q < p$ is assumed, which under our assumptions is provided by Theorem 5.4. Here q will depend on the ball B_0 .

Note that some weak Harnack inequalities in [34], [35] and [36] need to be modified, taking into account the dilation constant λ from the Poincaré inequality, see Björn–Marola [13, Section 10]. This is reflected in the constant $\Lambda \geq 1$ in Theorem 10.2 in the following way: Several of the weak Harnack inequalities in [6] contain a requirement that $50\lambda B \subset \Omega$. (The factor 50 is not the same in all the papers.) For a fixed ball $B_0 \subset X$, we let C_{PI} and λ be the constants in the p -Poincaré inequality (or q -Poincaré inequality) within $50B_0$, and C_μ be the doubling constant within $50\lambda B_0$. The weak Harnack inequality then holds for every ball $B \subset B_0$ provided that $50\lambda B \subset \Omega$ and with a constant depending only on C_{PI} , λ , C_μ and p (and q).

Proof. The arguments in [34, Section 4 and Theorem 5.1], [35, Section 5] and [36] are all local, so local assumptions are enough. They do rely on a better q -Poincaré inequality but a suitable version is provided by Theorem 5.3, as continuity is a local property. For the lsc-regularity of quasisuperminimizers also a Lebesgue point result is needed, which is justified by Theorem 7.1.

For the weak Harnack inequalities it is explained above how the semilocal dependence on the constants is achieved. \square

Apart from that the parameters may only be semilocal, the results in Chapters 7–14 in [6] all hold, with the exception of the Liouville theorem, which we look at in Example 10.3 below. This is because all the other results are of a local or semilocal nature, i.e. either in a bounded domain or concerning a local or semilocal property. In particular, in addition to the interior regularity in Theorem 10.2, one can prove various convergence results, minimum and maximum principles, solve the Dirichlet and the obstacle problem on bounded domains and obtain boundary regularity and resolutivity for suitable boundary data.

On the other hand, for results of a global nature, such as the Dirichlet problem on unbounded domains (as in Hansevi [23], [24]) or global singular functions (as in Holopainen–Shanmugalingam [28]), it is far from clear whether they hold under (semi)local assumptions. Thus, it is precisely when studying “global” properties that it is really interesting to know if the results hold with only local assumptions, possibly uniform ones, or if perhaps other properties of the space play a vital role. As an example of a “global” result, we will now have a look at the Liouville theorem, and show that it does not hold in the generality considered here, not even under local uniform assumptions, cf. Section 6.

Example 10.3. Let $d\mu = w dx$ on \mathbf{R} , where $\alpha \in \mathbf{R}$ and

$$w(x) = \begin{cases} 1, & |x| \leq 1, \\ |x|^\alpha, & |x| \geq 1. \end{cases}$$

The measure μ is globally doubling and supports a local uniform 1-Poincaré inequality, and thus a semilocal 1-Poincaré inequality. (And a global p -Poincaré inequality if and only if $\alpha < p - 1$.) A simple calculation shows that a function u is p -harmonic on (\mathbf{R}, μ) if and only if there is a constant c such that $u'(x) = cw(x)^{1/(1-p)}$. If $\alpha > p - 1$, this gives bounded nonconstant p -harmonic functions on (\mathbf{R}, μ) , namely

$$u(x) = \begin{cases} cx + b, & |x| \leq 1, \\ b + c \left(\frac{1}{\beta} + 1 - \frac{1}{\beta|x|^\beta} \right) \operatorname{sgn} x, & |x| \geq 1, \end{cases} \quad \text{where } \beta = \frac{\alpha - (p - 1)}{p - 1} > 0$$

and $b, c \in \mathbf{R}$ are arbitrary constants. This shows that the Liouville theorem does not hold under semilocal assumptions, nor under uniform local ones.

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